On the expression of measurement uncertainty for complex quantities with unknown phase

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Abstract

Simple expressions are obtained for the measurement uncertainty of complex quantities when no information about phase is available. Three different cases are considered: a known magnitude, an upper bound on the magnitude and an estimate of the magnitude with an associated uncertainty. An expression is also given for the uncertainty of a product when there is no information about the phase of the factors. These results are applied to simple examples of power, attenuation and impedance measurements. The performance of uncertainty calculations is checked by observing the coverage of uncertainty statements achieved under simulated measurement conditions.

Keywords: measurement uncertainty, complex quantity, mismatch uncertainty

1. Introduction

In radio frequency and microwave measurement (henceforth RF measurements), there are occasions when it is not possible to estimate the phase of a complex quantity. Traditionally, the most extreme values that might arise in a 'worst-case' measurement were used to evaluate the uncertainty. However, in a 1981 paper, Harris and Warner described an assessment of measurement uncertainty based on a probabilistic model [1]. Harris and Warner's treatment has gained wide acceptance and is compatible with the methods of propagating uncertainty described in the *Guide to the expression of uncertainty in measurement* (GUM) [2], published later in 1993.

Harris and Warner considered that, under certain circumstances, the phase of a complex measurement parameter with known magnitude could be considered as a random quantity distributed uniformly over 360 degrees. They showed that the real component of this random phasor follows an arcsine, or U-shaped, distribution bounded by the most extreme values. This is a useful result for the calculation of uncertainty for measurements of real-valued quantities, such as RF power and attenuation, where arcsine distributions can be associated with estimates of the components of complex influence quantities.

Complex quantities, such as reflection coefficients and complex scattering parameters, are also routinely measured. So there is interest in ways of evaluating the uncertainty in that case too. The RF community has developed bivariate extensions of the GUM's *Law of Propagation of Uncertainty* (LPU) that can be applied to measurement equations for complex quantities [3, 4, 5]. These extensions require the uncertainty of influence quantities to be expressed in a suitable form. There are also instances where 'scalar' measurements of a complex quantity are made, or where the information reported about the measurement of a complex quantity is equivalent to a scalar measurement. It is important to express this information in a form suitable for further uncertainty calculations, in keeping with the principle of *transferability*, which is one of the cornerstones of the GUM methods [2, §0.4]. Until recently, the problem of ignorance of phase had not been revisited in the context of a complex measurement. However, it turns out that a representation in the complex plane is simple and insightful and that there are several different cases worthy of consideration.

A recent paper looked at the common situation where only the phasor magnitude is known, as well as the situation where only an upper bound for magnitude is known, and gave simple matrix expressions for the uncertainty that can be used with the bivariate forms of the LPU [6, §2]. When these cases are considered in the complex plane, the distribution associated with unknown quantity is a

uniform ring or a uniform disk, respectively. The uniform ring is essentially the two-dimensional representation of same the problem considered by Harris and Warner; the uniform disk, which is also of practical interest, does not seem to have been considered before.

This paper discusses two more cases of practical interest. The first is the situation where the magnitude of a quantity has been measured, with some known uncertainty, but no phase information is available. The second addresses the uncertainty of a product of complex quantities, all with unknown phase. The network equations that describe RF measurements often include such products. However, standard GUM methods of uncertainty propagation do not work when the phases are unknown and so a different approach is needed.

When evaluating uncertainty by GUM methods, an assumption is made that the combined measurement error in a result has an approximately Gaussian distribution [2, §G.2.3]. This allows an expanded uncertainty interval to be calculated with a specified level of confidence, or coverage probability. However, the distributions associated with the unknown phase problems are not at all Gaussian-like. So the Gaussian approximation is a legitimate concern when quantities with unknown phase influence a measurement. Several scenarios are investigated here, using a simulation method to test the coverage of uncertainty statements. We find, reassuringly, that the coverage observed is close to, or above, nominal.

The following section summarises the expressions for uncertainty in the four cases mentioned above, section 3 then applies these expressions to simple measurement situations. Section 4 evaluates the performance of uncertainty calculations in measurement scenarios taken from §3. Section 5 gives a brief discussion about the interpretation of the uncertainties involved.

1.1. Notation

There is a convention in statistical writing to use uppercase roman characters for random variables and lowercase characters to denote estimates. In many measurement domains that convention clashes with standard notations. So, in the main article the distinction between fixed quantities, estimates and random variables is determined by the context, while in the appendices, the convention is largely followed. Real-valued quantities and quantity estimates are written in plain italic font, like X or x. Complex-valued quantities and quantity estimates are written in bold italic font, like $\boldsymbol{X}$ or $\boldsymbol{x}$. Greek characters are not italicised, but are written in plain style when representing real values and in bold when representing a complex values, e.g., Γ and $\boldsymbol{\Gamma}$. The imaginary unit j, where $\mathrm{j}^2 = -1$, is used, e.g.,

$\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}\, x_{\mathrm{im}}$ (note too, the use of subscripts identifying the real and imaginary components).

2. Complex quantities with unknown phase

The measurement of a complex quantity $\boldsymbol{X}$ provides estimates, x_{re} and x_{im}, of the real and imaginary components. When the phase of is unknown, an *a priori* assessment of measurement errors associates equal uncertainties $u(x_{\mathrm{re}}) = u(x_{\mathrm{im}}) = u$ with the real and imaginary component estimates, which are also considered uncorrelated. The full variance-covariance matrix representation of uncertainty then has a simple diagonal form [5]

$$\begin{bmatrix} u^2 & 0 \\ 0 & u^2 \end{bmatrix} .$$

To simplify the presentation below, scalar expressions corresponding to u will be given for the uncertainty of a complex quantity, rather than a matrix. It should be remembered, however, that the same uncertainty applies to both the real and imaginary components and that there is no correlation between the component estimates.

In general, a statement of uncertainty for a complex quantity defines a region in the complex plane that is likely to contain the quantity of interest. The shape of the region depends on the relative size of the component uncertainties as well as any correlation between them. When the uncertainties are equal, and when the correlation is zero, the shape of the uncertainty region is a circle.

The subsections below summarize some uncertainty expressions for situations that often arise in RF measurements.

2.1. Known magnitude (ring distribution)

When the magnitude of a complex quantity is known $|\boldsymbol{\Gamma}| = a$, but the phase is not, values that could be attributed to $\boldsymbol{\Gamma}$ lie on a circle of radius a, centred on the origin in the complex plane. Somewhat disconcertingly the best estimate of $\boldsymbol{\Gamma}$ for the purpose of uncertainty calculation is zero, at the geometrical centre of the locus of possible values. The associated standard uncertainty is [6, §2]

$$u = \frac{a}{\sqrt{2}} . \tag{1}$$

Readers may recognise this as the uncertainty associated with an arcsine distribution, of half-width a, as described by Harris and Warner [1, eq-14].

However, the familiar result involving the arcsine distribution applies to a real-valued quantity, such as attenuation or mismatch. Here, u applies to *both* the real and imaginary component estimates.

2.2. Bounded magnitude (disk distribution)

When the magnitude is bounded $|\mathbf{\Gamma}| \leq a$, but the phase is unknown, values that could be attributed to $\mathbf{\Gamma}$ are uniformly distributed on a disk of radius a at the origin. The best estimate of $\mathbf{\Gamma}$ is again zero and the associated standard uncertainty is [6, §2]

$$u = \frac{a}{2} \,. \tag{2}$$

2.3. Magnitude estimate

When the magnitude has been measured $|\mathbf{\Gamma}| \approx a$, with a standard uncertainty $u(a)$, the values that could be attributed to $\mathbf{\Gamma}$ are distributed in a diffuse ring that peaks at radius a and has radial symmetry around the origin. The associated standard uncertainty is (see Appendix A)

$$u = \sqrt{\frac{a^2}{2} + u^2(a)} \,. \tag{3}$$

In the limit of $a \gg u(a)$ equation (3) is equivalent to equation (1) for a uniform ring and when $a \ll u(a)$ the standard uncertainty u is equivalent to the measurement uncertainty $u(a)$. This later situation is not uncommon in RF metrology. For example, the reflection coefficient of a precision load can be very close to zero.

Unlike the previous cases, the distribution associated with possible values of $\mathbf{\Gamma}$ does not have a simple geometry. However, the standard uncertainty of a uniform annulus, with internal radius $a - \sqrt{2}u(a)$ and external radius $a + \sqrt{2}u(a)$, yields the same standard uncertainty [6, §2] and so could be used as a helpful intuitive representation of this case.

2.4. Product of estimates

Equations describing RF networks frequently contain sums of products and often, in practice, no phase information is available about the factors involved. This presents a problem for uncertainty calculation, because the GUM method does not propagate uncertainty from a product when both factors are zero-valued.

For the purposes of the uncertainty calculation, the product $\boldsymbol{G} = \mathbf{\Gamma}_1\mathbf{\Gamma}_2$ should be treated as single independent quantity. The standard uncertainty associated with

$\boldsymbol{G}$ is shown in Appendix B to be

$$u(\boldsymbol{G}) = \sqrt{2}u(\boldsymbol{\Gamma}_1)u(\boldsymbol{\Gamma}_2) \,, \tag{4}$$

where $u(\boldsymbol{\Gamma}_1)$ and $u(\boldsymbol{\Gamma}_2)$ are the standard uncertainties associated with the factors.

Note that equation (4) is needed when both factor estimates are zero. If one complex factor has been measured,‡ the phase of the product is still undetermined, but GUM methods could be applied to evaluate an uncertainty (although only the uncertainty component associated with the unknown phase quantity will be propagated).

3. Examples

3.1. Power measurement example

The power output of an RF signal generator is a common type of measurement. In a simple scenario, the measurement equation is

$$P_{\mathrm{g}} = MP_{\mathrm{i}} \,, \tag{5}$$

where P_{i} is the net RF power available to a power sensor (the difference between incident and reflected power) and

$$M = |1 - \boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}|^2 \,, \tag{6}$$

which is sometimes referred to as the mismatch error. Mismatch can, in principle, be corrected when the complex reflection coefficient of the signal generator output, $\boldsymbol{\Gamma}_{\mathrm{g}}$, and the power sensor input, $\boldsymbol{\Gamma}_{\mathrm{s}}$ are known. In practice, however, phase information about $\boldsymbol{\Gamma}_{\mathrm{g}}$ and $\boldsymbol{\Gamma}_{\mathrm{s}}$ is often unavailable. Both $|\boldsymbol{\Gamma}_{\mathrm{g}}|$ and $|\boldsymbol{\Gamma}_{\mathrm{s}}|$ will be small, so $M \approx 1$ and the associated uncertainty must account for the unknown phase. The uncertainty due to mismatch error can dominate the uncertainty budget in such cases.

Considering the the real and imaginary components of $\boldsymbol{G} = \boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}$, we may write

$$M = 1 - 2G_{\mathrm{re}} + G_{\mathrm{re}}^2 + G_{\mathrm{im}}^2 \,.$$

Differentiating this with respect to G_{re} and G_{im}, and remembering that $\boldsymbol{G} \approx 0$, we find that there is sensitivity only to the the real component

$$\frac{\partial M}{\partial G_{\mathrm{re}}} = 2(G_{\mathrm{re}} - 1) \approx -2 \tag{7}$$

$$\frac{\partial M}{\partial G_{\mathrm{im}}} = 2G_{\mathrm{im}} \approx 0 \,, \tag{8}$$

‡ this might happen in power measurements where the reflection coefficient of a power sensor can be measured relatively easily, but it is difficult to measure the reflection coefficient of the signal generator

so the standard uncertainty of the mismatch error

$$u(M) = 2u(G_{\mathrm{re}}) \ .$$

The contribution from each factor can be made explicit using equation (4)

$$u(M) = 2\sqrt{2}\, u(\Gamma_{\mathrm{s{\cdot}re}}) u(\Gamma_{\mathrm{g{\cdot}re}})$$

and then by considering different combinations of ring and disk distributions we can obtain the conventional Harris and Warner treatment of this problem, as well as two alternatives with lower uncertainties. First, when $|\boldsymbol{\Gamma}_{\mathrm{s}}|$ and $|\boldsymbol{\Gamma}_{\mathrm{g}}|$ are assumed known, the standard uncertainties in the real components are $u(\Gamma_{\mathrm{s{\cdot}re}}) = |\boldsymbol{\Gamma}_{\mathrm{s}}|/\sqrt{2}$ and $u(\Gamma_{\mathrm{g{\cdot}re}}) = |\boldsymbol{\Gamma}_{\mathrm{g}}|/\sqrt{2}$ and the standard uncertainty is

$$u(M) = \sqrt{2}\, |\boldsymbol{\Gamma}_{\mathrm{s}}||\boldsymbol{\Gamma}_{\mathrm{g}}| \ .$$

This is the same as the Harris and Warner result. However, when one magnitude is known and one is bounded the uncertainty is reduced by a factor of $\sqrt{2}$, giving

$$u(M) = |\boldsymbol{\Gamma}_{\mathrm{s}}||\boldsymbol{\Gamma}_{\mathrm{g}}|$$

and when both magnitudes are bounded the uncertainty is half the conventional value

$$u(M) = \frac{|\boldsymbol{\Gamma}_{\mathrm{s}}||\boldsymbol{\Gamma}_{\mathrm{g}}|}{\sqrt{2}} \ .$$

3.2. Attenuation measurement example

An attenuation measurement can be made by taking the ratio of two power measurements: one with a direct connection between a power sensor and a signal source, the other with a device under test (DUT) inserted between the source and sensor. In a simple scenario (see Appendix C), four complex quantities influence the measurement result: the reflection coefficients $\boldsymbol{\Gamma}_{\mathrm{g}}$ and $\boldsymbol{\Gamma}_{\mathrm{s}}$, of the signal generator and the power sensor, and the reflection coefficients associated with the entrance and exit ports of the DUT, denoted $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{22}$.

Following the analysis in Appendix C, an equation for attenuation in terms of the power ratio R is

$$A_{\mathrm{DUT}} \approx -10 \log_{10} R - 8.686\, (A_{\mathrm{re}} - B_{\mathrm{re}} - C_{\mathrm{re}} - D_{\mathrm{re}}) \ , \tag{9}$$

where A_{re}, B_{re}, C_{re} and D_{re} are the real components of

$$\boldsymbol{A} = \boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}$$
$$\boldsymbol{B} = \boldsymbol{S}_{11}\boldsymbol{\Gamma}_{\mathrm{g}}$$
$$\boldsymbol{C} = \boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\mathrm{s}}$$
$$\boldsymbol{D} = S^2_{21{\cdot}\mathrm{nom}}\boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}$$

and $S_{21\cdot\mathrm{nom}}$ is the nominal magnitude of the transmission coefficient of the DUT. In the context of attenuation measurement, the term $8.686\,(A_{\mathrm{re}} - B_{\mathrm{re}} - C_{\mathrm{re}} - D_{\mathrm{re}})$ may also be referred to as a mismatch error.

There is usually no information about the phases of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ so these complex terms must be estimated as zero. The mismatch term is therefore zero and we treat the different uncertainty components as products using equation (4)§

$$
\begin{aligned}
u(A_{\mathrm{re}}) &= \sqrt{2}\, u_{\mathrm{s}} u_{\mathrm{g}} \\
u(B_{\mathrm{re}}) &= \sqrt{2}\, u_{11} u_{\mathrm{g}} \\
u(C_{\mathrm{re}}) &= \sqrt{2}\, u_{22}\, u_{\mathrm{s}} \\
u(D_{\mathrm{re}}) &= \sqrt{2} S_{21\cdot\mathrm{nom}}^2\, u_{\mathrm{s}} u_{\mathrm{g}}\ .
\end{aligned}
$$

As in §3.1, we obtain the same result as Harris and Warner when all the uncertainties are associated with ring distributions. However, the uncertainty will be reduced if some uncertainties can be associated with disk distributions.

3.3. Vector network analyzer example

RF measurements of complex reflection and transmission coefficients are generally made with an instrument called a vector network analyzer (VNA). These measurements are subject to relatively large systematic errors, which are corrected as part of the measurement procedure (usually by instrument software). The remaining residual errors after such adjustments make an important contribution to the measurement uncertainty.

Fairly well-known methods of estimating the magnitude of residual VNA errors are described, for example, in reference [7]. Only magnitude information is obtained in this way, however, so the terms arising in the corresponding measurement equations should be treated as quantities with unknown phase.

A recent review of reference [7] clarified some details relating to the measurement equations and derived GUM-compatible expressions for the components of uncertainty for some basic types of VNA measurement [8]. The simplest case is when a one-port measurement of a complex reflection coefficient $\boldsymbol{\Gamma}$ is made. There are three residual systematic errors in such a measurement, denoted $\boldsymbol{D}$, $\boldsymbol{M}$ and $\boldsymbol{T}$. The combined uncertainty of $\boldsymbol{\Gamma}$ can be expressed as‖

$$
u(\Gamma) = \left[u_D(\Gamma)^2 + u_T(\Gamma)^2 + u_M(\Gamma)^2\right]^{1/2}\ . \tag{10}
$$

§ For simplicity, subscripts on the right-hand terms are associated with the quantities $\boldsymbol{\Gamma}_{\mathrm{g}}$, $\boldsymbol{\Gamma}_{\mathrm{s}}$ and $\boldsymbol{S}_{11}$. So, for example, u_{s} is the standard uncertainty associated with an estimate of $\boldsymbol{\Gamma}_{\mathrm{s}}$.

‖ Additional terms occur of course in a full uncertainty budget. The expression given here deals only with the systematic instrument errors.

The terms in this expression are

$$u_D(\Gamma) = u(D) \,, \tag{11}$$

$$u_T(\Gamma) = |\boldsymbol{\Gamma}_\mathrm{m}| u(T) \,, \tag{12}$$

$$u_M(\Gamma) = |\boldsymbol{\Gamma}_\mathrm{m}|^2 u(M) \,, \tag{13}$$

where $u(D)$, $u(T)$ and $u(M)$ are standard uncertainties associated with the estimates $\boldsymbol{D} \approx 0$, $\boldsymbol{M} \approx 0$ and $\boldsymbol{T} \approx 1$ and $|\boldsymbol{\Gamma}_\mathrm{m}|$ is the magnitude of the measured reflection coefficient. Reference [7] associates an arcsine distribution with these residual uncertainties. However, actual estimates can be rather crude and a disk distribution might better represent some of the information available, which would reduce the uncertainty. For instance, if an instrument specification is used to estimate the magnitude of the residual tracking error $|\boldsymbol{T}|$ [7, 6.2.4.1], the uniform disk distribution is arguably a more appropriate representation of the information available.

4. Uncertainty calculation performance

Measurement uncertainty calculations are based on a model of the measurement errors that contribute to a result. GUM uncertainty calculations make the assumption that the combined measurement error is approximately Gaussian. This allows an expanded uncertainty interval to be calculated for a specific coverage probability, or level of confidence. However, the influence quantity estimates used as inputs to GUM calculations need not be associated with Gaussian errors, which can lead to problems when non-Gaussian errors dominate a measurement uncertainty budget. When this happens, the stated level of confidence may not be accurate.

In this work, the uniform ring and disk distributions associated with complex quantity estimates are non-Gaussian. So if errors having these distributions play a prominent role in an uncertainty budget, there is good reason to question the accuracy of GUM uncertainty calculations. In this section, the examples discussed in §3 are used as scenarios to test the coverage probability of uncertainty calculations.

To assess the performance of a particular method of uncertainty calculation, the measurement procedure is simulated repeatedly to generate independent sets of input data for an uncertainty calculation. The uncertainty statement obtained for each data set can be compared with the simulation parameters, which are known (i.e., the measurand is known), and the proportion of statements that include the measurand can be recorded and compared with the nominal coverage probability. For example, if $N = 10^5$ measurement simulations are performed and the uncertainty statement in 9.5×10^4 is found to contain the measurand, then

the observed 95% *success-rate* is an estimate of the coverage probability for that procedure. Due to the finite number of simulations, there will be some uncertainty in this estimate of coverage probability. For $N = 10^5$, the standard deviation in the observed success-rate is about 0.1% for a procedure with roughly 95% coverage probability.

This method of checking uncertainty calculations has been described in the context of a simple RF measurement [9, 10], and was applied recently to several examples from the GUM [11].

4.1. Mismatch and power measurement

In the scenario described in §3.1, the measured power depends on the complex reflection coefficients of the RF generator and power sensor. We simulate the power P_i indicated by a measurement (the value displayed by the power meter) using the following model

$$P_{\mathrm{i}} = \frac{P_{\mathrm{g}}}{|1 - \boldsymbol{G}|^2} + X_{\mathrm{n}} \ ,$$

where the product $\boldsymbol{G} = \boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}$ is treated as a random complex quantity and X_{n} represents additive system noise. This expression was used to generate values of P_{i} by assigning random number generators to $\boldsymbol{G}$ and X_{n} and fixing the value of $P_{\mathrm{g}} = 1$ mW. For each value of P_{i}, a GUM uncertainty calculation was performed using the standard uncertainty associated with system noise, $u(X_{\mathrm{n}})$, and the standard uncertainty associated with mismatch, $u(M) = 2u(\Gamma_{\mathrm{re}})$. The latter depends on the information available about $\boldsymbol{\Gamma}_{\mathrm{s}}$ and $\boldsymbol{\Gamma}_{\mathrm{g}}$. For example, if $|\boldsymbol{\Gamma}_{\mathrm{s}}|$ and $|\boldsymbol{\Gamma}_{\mathrm{g}}|$ are both known, $u(M) = \sqrt{2}\,|\boldsymbol{\Gamma}_{\mathrm{s}}||\boldsymbol{\Gamma}_{\mathrm{g}}|$. Finally, the result is checked to see whether P_{g} is contained within the calculated expanded uncertainty interval. A large number of experiments were simulated after choosing $u(X_{\mathrm{n}})$ and a particular distribution to represent the product $\boldsymbol{G} = \boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}$, which fixes $u(M)$.

The observed success-rates are reported in Table 1. The different values of $u(M)$ correspond to different combinations of factors in $\boldsymbol{G} = \boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}$. The radius associated with the ring, disk or ring-estimate (u-ring) distributions was $a = 0.1$ in each case and $u(a) = a/3$ was used in the case of the ring-estimate. The columns labeled 'Noise' report the observed coverage for different values of the system noise $u(X_{\mathrm{n}})$.

The results in column $u(X_{\mathrm{n}}) = 0.00$ model the measurement with just a mismatch error. This is really a worst-case scenario, consisting of a single, dominant, non-Gaussian error. Some departure from nominal performance is expected under these conditions. The results obtained show that uncertainty statements are conservative, with essentially full coverage in every case except the product of two

Distribution	Mismatch	Noise $u(X_\mathrm{n})$ mW					
$\boldsymbol{G} = \boldsymbol{\Gamma}_\mathrm{s}\boldsymbol{\Gamma}_\mathrm{g}$	$u(M)$	0.00	0.01	0.03	0.1	0.3	1.0
ring × ring	0.141	100.0	100.0	99.4	96.2	95.5	95.9
u-ring × u-ring	0.156	100.0	100.0	99.8	97.0	95.8	96.0
disk × ring	0.100	99.0	98.7	97.4	95.4	95.4	95.5
disk × disk	0.071	94.9	95.0	95.0	95.1	95.2	95.1

Table 1. Observed success-rates (in %) for $N = 10^5$ simulations. The standard deviation in the success-rate for $N = 10^5$ is approximately 0.1%. The measurand is a power level of 1 mW.

disk distributions. The expanded uncertainty interval is wide enough to always contain the measurand. So, the assumption of a Gaussian error in the result has allowed for a broader distribution of error than required.

The simulation reported in row 4 of column $u(X_\mathrm{n}) = 0.00$ models Γ as the product of two uniform-disk distributed errors. The results in this case are nearly nominal, suggesting that distribution of error for the product is sufficiently Gaussian-like for GUM calculations to achieve nominal coverage. The distribution of the product of two disk distributions will peak at the origin and drop away to zero at a radius of a^2.

The other columns in Table 1 report coverage results for increasing levels of system noise. The trends are predictable. Coverage in the clearly non-Gaussian cases stays above nominal until the system noise is comparable to the mismatch error (column $u(X_\mathrm{n}) = 0.1$). Then, as the Gaussian noise increases, coverage tends towards 95%. These results indicate that good uncertainty calculation performance can be expected in most practical power measurements, where mismatch may be an important source of uncertainty, but is unlikely to account for more than half the total uncertainty budget. Moreover, the effect of a particularly dominant mismatch error would only lead to conservative uncertainty statements that do not over-state the accuracy of the measurement procedure.

4.2. Attenuation

The attenuation measurement described in §3.2 involves a pair of power measurements. The quantity intended to be measured is the attenuation of a device $A_\mathrm{DUT} = -10\log_{10}|S_{21}|^2$, which is a function of the device transmission coefficient $|S_{21}|$. The ratio of power readings also depends on mismatch factors associated with each measurement

$$R = \frac{P_1}{P_0} = S_{21}^2 \frac{|M_1|^2}{|M_0|^2} \,,$$

where P_1 is a measurement with a device connected between the signal generator and the power sensor and P_0 is without the device. The mismatch term

$$M_0 = 1 - \mathbf{\Gamma}_\mathrm{s}\mathbf{\Gamma}_\mathrm{g}$$

is associated with P_0 and

$$M_1 = (1 - \boldsymbol{S}_{11}\mathbf{\Gamma}_\mathrm{g})(1 - \boldsymbol{S}_{22}\mathbf{\Gamma}_\mathrm{s}) - S_{21}^2\mathbf{\Gamma}_\mathrm{s}\mathbf{\Gamma}_\mathrm{g}$$

is associated with P_1.

To simulate the ratio of P_0 and P_1, for a device with $S_{21} = 0.01$, random values of M_0 and M_1 were generated and some system noise was also added. The power measurements would normally be made on different instrument ranges, so noise was set at a fraction of the actual power level and Gaussian random number generators were used to simulate a noise component in each reading of P_0 and P_1. The complex products $\boldsymbol{S}_{11}\mathbf{\Gamma}_\mathrm{g}$, $\boldsymbol{S}_{22}\mathbf{\Gamma}_\mathrm{s}$ and $\mathbf{\Gamma}_\mathrm{s}\mathbf{\Gamma}_\mathrm{g}$ were treated as independent quantities and simulated by random number generators for the appropriate type of distribution. For simplicity, the same type of distribution was associated with each product and the characteristic radius was set equal 0.05.

The observed success-rates are shown in Table 2. The equation that would be used for data processing in this measurement is (see Appendix C)

$$A_\mathrm{DUT} \approx= -10\log_{10} R - 8.686(A_\mathrm{re} - B_\mathrm{re} - C_\mathrm{re} - D_\mathrm{re}) \ ,$$

where A_re, B_re, C_re and D_re are the real components of complex product terms in the mismatch expressions. The standard uncertainty u, associated with each distribution is reported in the second column of the table. This was used to evaluate the uncertainty of A_re, B_re, C_re and D_re. Measurement system noise is reported in the table as a percentage of power reading under the heading 'Noise'. The noise component in each power reading was propagated to A_DUT using standard GUM methods.

Mismatch		Noise $u(X_\mathrm{n})$ (%)			
Distribution	u	0	2	5	10
ring × ring	0.0354	95.1	95.3	95.3	95.0
disk × ring	0.0250	95.3	95.5	95.1	94.8
disk × disk	0.0177	95.0	94.8	95.1	94.7

Table 2. Observed success-rates (in %) for $N = 10^5$ simulated attenuation measurements. The standard deviation in the success-rate for $N = 10^5$ is approximately 0.1%.

In Table 2, the results in column $u(P_\mathrm{i}) = 0$ show that uncertainty calculations perform well, even when the system noise is zero. This contrasts with the power

measurements in §4.1, where the results for mismatch modeled as a ring or a disk distribution both gave much higher coverage. The difference here is that there are a number of independent mismatch error terms that contribute to the combined measurement error. The near-nominal success-rates indicate that the distribution of the combined measurement error is already a satisfactory approximation to a Gaussian distribution for the purposes of GUM uncertainty calculations.

As the system noise level is increased, we would not expect coverage performance to deteriorate. However, at the highest noise level, when $u(X_\mathrm{n}) = 10\,\%$, there is a perceptible drop in coverage below nominal. In this situation, the system noise is a much more important source of errors than the effects of mismatch, so the non-linearity of the term $-\log_{10} P_1/P_0$, from a power ratio to attenuation, will be start to distort the distribution of measurement error.

4.3. Complex reflection coefficient

The VNA measurement of a complex reflection coefficient $\boldsymbol{\Gamma}$ described in §3.3 is subject to three complex residual errors $\boldsymbol{D} \approx 0$, $\boldsymbol{M} \approx 0$ and $\boldsymbol{T} \approx 1$. To simulate the reflection coefficient indicated by a VNA, $\boldsymbol{\Gamma}_\mathrm{m}$, random values for $\boldsymbol{D}$, $\boldsymbol{M}$ and $\boldsymbol{T}$ were drawn from an appropriate type of distribution and some system noise added. The equation used to simulate data was

$$\boldsymbol{\Gamma}_\mathrm{m} = \boldsymbol{D} + \frac{\boldsymbol{T}\boldsymbol{\Gamma}}{1 - \boldsymbol{T}\boldsymbol{M}\boldsymbol{\Gamma}} + \boldsymbol{X}_\mathrm{n} \,, \tag{14}$$

where $\boldsymbol{X}_\mathrm{n}$ represents bivariate Gaussian random noise and $\boldsymbol{\Gamma}$ is the measurand. For simplicity, the same type of distribution was associated with $\boldsymbol{D}$, $\boldsymbol{M}$ and $\boldsymbol{T}$ and the radius of the distribution was set at $a = 0.01$. The complex random variable $\boldsymbol{X}_\mathrm{n}$ was generated with the same variance in the real and imaginary components and zero covariance.

In processing the data from these measurement simulations, we have included an uncertainty component for the system noise $u_X(\Gamma) = u(X_\mathrm{n})$, the standard deviation of one component of $\boldsymbol{X}_\mathrm{n}$, in addition to the three components of uncertainty described in equations (11)–(13). These four components are added in quadrature (c.f., equation (10)) to obtain a standard uncertainty associated with the measurement result. The region of uncertainty in this case is a circle in the complex plane [8], so a successful uncertainty calculation is one that generates a circular region around $\boldsymbol{\Gamma}_\mathrm{m}$ that includes $\boldsymbol{\Gamma}$.¶

The success-rates for simulated measurements of a low-reflection component, with $\Gamma = 0.05 + \mathrm{j}0.01$, are shown in Table 3. Simulations for medium and highly

¶ The coverage factor required to scale the radius of the uncertainty region is $k_2 = \chi_{2,0.95}$, where $\chi_{2,0.95}$ is the 95$^{\mathrm{th}}$ percentage point of the chi distribution with 2 degrees of freedom [8].

reflecting components gave essentially the same results and so are not reported here. The observed success-rates show that when the system noise is small the uncertainty statements are conservative, but that coverage quickly trends towards nominal as the system noise increases. Note that smaller uncertainty statements will be generated when disk distributions are associated with residual errors, rather than ring distributions.

Residuals		Noise $u(X_{\mathrm{n}})$				
Distribution	u	0.001	0.005	0.01	0.05	0.10
ring	0.007	100.0	98.1	95.7	95.0	95.0
disk	0.005	99.9	96.0	95.2	95.0	95.0

Table 3. Observed success-rates (in %) for $N = 10^5$ simulations. The standard deviation in the success-rate for $N = 10^5$ is approximately 0.1%. The measurand $\mathbf{\Gamma}$ and the standard uncertainties u and $u(X_{\mathrm{n}})$ are dimensionless.

One of the assumptions made in the simulations is that the real and imaginary components of the system error have the same variance. This is in keeping with the assumptions made in [7, 8], on which the data processing is based. However, different amounts of noise in the real and imaginary components will affect coverage. Table 4, reports a series of simulations where the variance of the real component of simulated system noise was set to twice the variance of the imaginary component. This causes the observed success-rates to a drop below nominal coverage.

If it is known that the distribution of system error is not isotropic, bivariate forms of uncertainty propagation (LPU) would be more appropriate for the data processing [3, 5]. We would expect these methods to restore nominal coverage under conditions where the system noise is dominant, and the uncertainties associated with $\boldsymbol{D}$, $\boldsymbol{M}$ and $\boldsymbol{T}$ can be easily used in such calculations (see §2 and [6]).

Residuals		Noise $u(X_{\mathrm{n}})$				
Distribution	u	0.001	0.006	0.012	0.061	0.122
ring	0.007	100.0	97.0	95.1	94.6	94.6
disk	0.005	99.8	95.4	94.9	94.6	94.6

Table 4. Observed success-rates (in %) for $N = 10^5$ simulations. The variance of the real component of system noise in these simulations is twice that of the imaginary component. The value reported for $u(X_{\mathrm{n}})$ is the square root of the mean variance.

5. Discussion and conclusions

This article has described several expressions for the measurement uncertainty of complex quantities when phase is unknown. In some important types of measurement, these type-B uncertainties produce a lower combined uncertainty than the arcsine distribution, which is currently the preferred representation for the unknown-phase problem. Simulation studies show that uncertainty calculations that use these alternative forms of uncertainty perform well in the context of typical RF measurements.

The expressions for uncertainty in equations (1)–(4) will be useful in a wide range of RF measurement uncertainty problems. However, it is important to appreciate the assumptions associated with these type-B uncertainties. There is an *a priori* assumption that there is an equal likelihood for the value of phase to be anywhere between 0 and 360 degrees. This cannot be true for a single measurement apparatus at a single operating frequency, where the phases, although unknown, are certainly well-defined, fixed, quantities. Harris and Warner considered this and commented that *".. uncertainty values have a statistical meaning over many calibrations using various combinations of apparatus."* [1, §2]. They also noted that a measurement might be *".. repeated with many different examples of properly designed apparatus, which differ in detailed design so that [phases will] differ from one apparatus to another..."*. So the interpretation of uncertainty statements, including the conventional use of the arcsine distribution, must be made carefully. They may be used to describe the expected statistical behaviour of a set of independent experiments [1, §2] and they may be useful for propagating uncertainty in further calculations, but the association of a statistical distribution with uncertain phase does not imply a constantly varying phase in a single measurement.[+]

Measurements are often used to characterise systems consisting of networks of interconnected components. The finite delays associated with these transmission sections, which are often neglected in analysis, correspond to phase shifts when frequency is varied and this may further justify the model of random phase in practice. For example, the usual schematic representation of a simple power measurement, as in §3.1, ignores the actual connection between generator and sensor. The delay associated with this connection will cause the value of the

[+] While this manuscript was being reviewed, we became aware of a recent study that measured the distribution of reflection coefficients in a small sample of RF power sensors, and some more complicated RF instruments [12]. The findings support the assumption of a uniformly random phase when a wide range of frequencies are pooled. However, ring-like distributions for $|\mathbf{\Gamma}|$ were not observed, instead magnitudes were widely spread. This suggests that the conventional Harris-Warner assumption of constant magnitude should be reconsidered in even quite simple measurement situations.

mismatch error to vary with frequency in a way that cannot be calculated unless the properties of the line are known. Over a broad range of frequencies a sample of different phases is generated and the assumption of a uniform distribution would probably be reasonable.

In conclusion, the uncertainty associated with complex measurement errors of unknown phase can be associated with probability distributions with radial symmetry around the origin. The covariance matrices for these distributions have a simple diagonal structure, allowing a single-parameter expression of the measurement uncertainty. The relatively simple geometries and the simple expressions of uncertainty offer useful and intuitive extensions to the conventional way of handling this type of problem, which associates an arcsine distribution with uncertainty when phase is unknown. The alternative expressions of uncertainty presented here can reduce the overall measurement uncertainty in some common types of RF measurement.

Acknowledgement

The author would like to thank R Willink, for his comments in relation to this work, as well as M Dobbert and J Gorin for useful discussions. The work was funded by the New Zealand Government as part of a contract for the provision of national measurement standards.

Appendix A. The uncertainty associated with a magnitude estimate

A simple model can be used to describe the situation where a measurement of $|\boldsymbol{\Gamma}|$ is obtained, with uncertainty $u(|\boldsymbol{\Gamma}|)$, but no information about phase is available.

We first consider an estimate $\boldsymbol{X}$ of $\boldsymbol{\Gamma}$ is subject to a bivariate Gaussian error. The covariance matrix of $\boldsymbol{X}$ is

$$E[(\boldsymbol{X}-\boldsymbol{\Gamma})^2] = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

and the matrix of second moments about the origin is

$$E[\boldsymbol{X}^2] = \begin{bmatrix} \Gamma_{\mathrm{re}}^2 + \sigma^2 & \Gamma_{\mathrm{re}}\Gamma_{\mathrm{im}} \\ \Gamma_{\mathrm{re}}\Gamma_{\mathrm{im}} & \Gamma_{\mathrm{im}}^2 + \sigma^2 \end{bmatrix},$$

where $\Gamma_{\mathrm{re}} = |\boldsymbol{\Gamma}|\cos\phi$ is the real component of $\boldsymbol{\Gamma}$ and $\Gamma_{\mathrm{im}} = |\boldsymbol{\Gamma}|\sin\phi$ is the imaginary component.

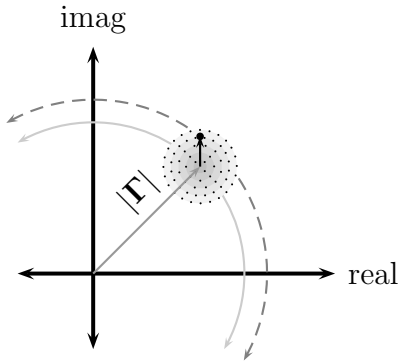

Figure A1. Error model for magnitude estimate

Next, we consider that a uniform phase error is added to $\boldsymbol{\Gamma}$, giving an estimate $\boldsymbol{X}_{\mathrm{mix}}$ of $\boldsymbol{\Gamma}$ that has an error distribution with radial symmetry around the origin. The elements of the covariance matrix $E[\boldsymbol{X}_{\mathrm{mix}}^2]$ can be obtained by averaging the elements of $E[\boldsymbol{X}^2]$ over $\phi \in [0, 2\pi]$, so after integration,

$$E[\boldsymbol{X}_{\mathrm{mix}}^2] = \begin{bmatrix} \frac{|\boldsymbol{\Gamma}|^2}{2} + \sigma^2 & 0 \\ 0 & \frac{|\boldsymbol{\Gamma}|^2}{2} + \sigma^2 \end{bmatrix}.$$

The square root of the diagonal elements of this matrix can be associated with the standard uncertainty of the real and imaginary components of $\boldsymbol{\Gamma}$ when a measurement of $|\boldsymbol{\Gamma}|$ is reported.*

* In the terminology of the GUM, the square root of the diagonal elements would be called a standard uncertainty and the letter u would be used.

Appendix B. Uncertainty associated with a product of zero estimates

We are interested in the situation where the factors of a complex product have been estimated in magnitude only. The variance-covariance matrix of the product $\boldsymbol{Z} = \boldsymbol{Z}_1\boldsymbol{Z}_2$, where

$$\boldsymbol{Z}_1 = X_1 + \mathrm{j}Y_1$$
$$\boldsymbol{Z}_2 = X_2 + \mathrm{j}Y_2$$

are random variables with zero means. The component variances $E(X_1^2) = E(Y_1^2) = \sigma_1^2$ and $E(X_2^2) = E(Y_2^2) = \sigma_2^2$♯ and the covariances between the real and imaginary components of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are all zero.

The product

$$\boldsymbol{Z} = \boldsymbol{Z}_1\boldsymbol{Z}_2 = (X_1X_2 - Y_1Y_2) + \mathrm{i}(X_1Y_2 + X_2Y_1)$$

has expectation zero, because

$$\begin{aligned} E(X_1X_2 - Y_1Y_2) &= E(X_1X_2) - E(Y_1Y_2) \\ &= E(X_1)E(X_2) - E(Y_1)E(Y_2) \\ &= 0 \end{aligned}$$

and similarly $E(X_1Y_2 + X_2Y_1) = 0$.

The variances of the real and imaginary components of $\boldsymbol{Z}$ are equal, because

$$\begin{aligned} &E\left[(X_1X_2 - Y_1Y_2)^2\right] \\ &\quad = E\left[X_1^2X_2^2 - 2X_1X_2Y_1Y_2 + Y_1^2Y_2^2\right] \\ &\quad = E(X_1^2X_2^2) - 0 + E(Y_1^2Y_2^2) \\ &\quad = \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_2^2 \\ &\quad = 2\sigma_1^2\sigma_2^2 \end{aligned}$$

and

$$E\left[(X_1Y_2 + X_2Y_1)^2\right] = 2\sigma_1^2\sigma_2^2 \ .$$

The covariance between the real and imaginary components of $\boldsymbol{Z}$ is zero

$$\begin{aligned} &E\left[(X_1X_2 - Y_1Y_2)(X_1Y_2 + X_2Y_1)\right] \\ &\quad = E(X_1^2X_2Y_2) - E(Y_2^2X_1Y_1) + E(X_2^2X_1Y_1) - E(Y_1^2X_2Y_2) \\ &\quad = 0 \ . \end{aligned}$$

So the variance-covariance matrix of $\boldsymbol{Z}$ is

$$2\begin{bmatrix} \sigma_1^2\sigma_2^2 & 0 \\ 0 & \sigma_1^2\sigma_2^2 \end{bmatrix}$$

♯ In the terminology of the GUM, σ_1 and σ_2 would be referred to as standard uncertainties and the letter u would replace σ.

and the standard uncertainty

$$u = \sqrt{2}\sigma_1\sigma_2$$

may be associated with estimates of the real and imaginary components of $\boldsymbol{Z}$.

Appendix C. Attenuation example

In the simple attenuation measurement in §3.2, the following expression describes the magnitude of the transmission coefficient

$$S_{21}^2 = R\left|\frac{1-\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}}{(1-\boldsymbol{S}_{11}\boldsymbol{\Gamma}_{\rm g})(1-\boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\rm s})-S_{21}^2\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}}\right|^2 , \qquad \text{(C.1)}$$

where R is a measured power ratio and $\boldsymbol{\Gamma}_{\rm g}$, $\boldsymbol{\Gamma}_{\rm s}$, $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{22}$ are all complex reflection coefficients with magnitudes close to zero.†† The quantity of interest, the attenuation, is defined as

$$A_{\rm DUT} = -10\,\log_{10} S_{21}^2 \ .$$

Certain approximations are commonly used to obtain a convenient expression for attenuation [1]. The magnitudes of all the reflection coefficient terms in this equation are small, so

$$\begin{aligned}
|1-\epsilon|^2 &= \left|\frac{1-\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}}{(1-\boldsymbol{S}_{11}\boldsymbol{\Gamma}_{\rm g})(1-\boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\rm s})-S_{21}^2\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}}\right|^2 \\
&= \left|\frac{1-\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}}{1-\boldsymbol{S}_{11}\boldsymbol{\Gamma}_{\rm g}-\boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\rm s}+\boldsymbol{S}_{11}\boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\rm g}\boldsymbol{\Gamma}_{\rm s}-S_{21}^2\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}}\right|^2 \\
&\approx \left|1-\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}+\boldsymbol{S}_{11}\boldsymbol{\Gamma}_{\rm g}+\boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\rm s}+S_{21}^2\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}\right|^2 ,
\end{aligned}$$

where $\boldsymbol{S}_{11}\boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\rm g}\boldsymbol{\Gamma}_{\rm s}$ has been dropped in the last line, being of second-order smallness compared to other terms. Now $|1-\boldsymbol{\epsilon}|^2 \approx 1-2{\rm Re}(\boldsymbol{\epsilon})$ when $|\boldsymbol{\epsilon}|^2 \ll |\boldsymbol{\epsilon}|$, so the transmission coefficient is conveniently expressed as

$$S_{21}^2 \approx R\,[1-2{\rm Re}(\boldsymbol{A}-\boldsymbol{B}-\boldsymbol{C}-\boldsymbol{D})]$$

with $\boldsymbol{A} = \boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}$, $\boldsymbol{B} = \boldsymbol{S}_{11}\boldsymbol{\Gamma}_{\rm g}$, $\boldsymbol{C} = \boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\rm s}$ and $\boldsymbol{D} = S_{21\cdot{\rm nom}}^2\boldsymbol{\Gamma}_{\rm s}\boldsymbol{\Gamma}_{\rm g}$. A nominal value for S_{21} is used in $\boldsymbol{D}$ because the actual value is the subject of measurement. Finally, using the approximation $\log_e(1+2x) \approx 2x$, for small x, this can be written as

$$\begin{aligned}
A_{\rm DUT} &\approx -10\log_{10} R - 2\frac{10}{\log_e 10}(A_{\rm re}-B_{\rm re}-C_{\rm re}-D_{\rm re}) \\
&\approx -10\log_{10} R - 8.686\,(A_{\rm re}-B_{\rm re}-C_{\rm re}-D_{\rm re}) \ .
\end{aligned}$$

†† Many other terms would influence a real measurement, such as: instrumental factors, linearity, leakage, etc. Our focus here is on mismatch error.

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