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VNA error models: Comments on EURAMET/cg-12/v.01

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Reference

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Summary

This report examines the error models referred to in the the EURAMET Calibration Guide cg-12/v.01 for evaluating VNA performance.

An analysis is carried out using a formulation of the *Law of Propagation of Uncertainty* developed recently for complex quantities. The standard equations for VNA raw-data correction are used to determine components of uncertainty associated with residual VNA calibration errors. A simple flow-graph analysis is given as a way of explaining the dominant errors.

The report identifies some opportunities to correct and clarify the Guide's current use of error models.

The report also comments that there is an opportunity to express the uncertainty of complex quantities and, in some cases, the uncertainty in magnitude and phase of results.

The revised version of this report contains a short Addendum with additional references.

This report is also available as ANAMET Report 051.

Contents

1 Introduction

The EURAMET Calibration Guide cg-12/v.01, for VNAs (henceforth, the Guide) offers concise practical information on how to assess the uncertainty of VNA measurements [1]. It introduces a few simple mathematical expressions that are said to represent error models for one-port and two-port measurements. The Guide does not discuss how these expressions were obtained. Nevertheless, the basis of its recommendations should be clearly understood for the maintenance, dissemination and possible extension of such guidelines.

This report takes conventional models for one-port and two-port measurements and derives expressions for the components of measurement uncertainty associated with VNA calibration errors. The analysis considers how errors in calibration constants propagate when the usual data-processing is applied. The general approach to uncertainty evaluation described in the Guide to the Expression of Uncertainty in Measurement (GUM) is followed [2]. A set of components of uncertainty are obtained by applying an extended form of the *Law of Propagation of Uncertainty*, recently developed for complex quantities by Yhland and Stenarson [3].

While the formal GUM-based approach gives a clear theoretical basis for statements about measurement uncertainty, a flow-graph analysis can be helpful to visualise the sources of measurement error. Simplified flow-graph analysis of each measurement situation is also given and it is shown that while the two approaches can be made to agree, they are the same. Flow-graph analysis naturally obtains an expression for the quantity observed in a measurement. This is actually a different from the error correction carried out during data-processing, so the two approaches are not necessarily equivalent.

The report devotes a section to each type of VNA measurement, at the end of which a brief 'summary' is given noting any changes that could simplify and clarify the Guide.

The report also includes a section discussing phase uncertainty and full uncertainty statements about complex quantities. It argues that the Guide could be extended to cover such cases and gives some relevant references to the scientific literature.

1.1 Notation

In this report real-valued quantities are shown in a plain font, like x or Γ, while complex quantities are shown in bold, like $\boldsymbol{x}$ or $\boldsymbol{\Gamma}$.

1.2 Propagation of uncertainty

The analysis of measurement uncertainty in this report uses a special formulation of the *Law of Propagation of Uncertainty* developed by Yhland and Stenarson recently (henceforth, the YS method) [3].

The method requires an analytic complex function describing the measurement. The following assumptions are also made:

- all influence quantities are independent (including real and imaginary components of the same complex quantity),

- uncertainties in the real and imaginary components of influence quantities are equal.

These are rather severe conditions. In effect, the three parameters normally associated with the uncertainty of a complex quantity (uncertainty in the real and imaginary components and the covariance between them) are reduced to just one. However, the Guide describes methods of estimating the magnitude of residual VNA errors and there is no way to ascribe any correlation between them. In this context the assumptions of the YS method may be acceptable.

The measurement function should be of the form

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots) \ ,$$

where $\boldsymbol{Y}$ is the measurand and $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots$ are influence quantities. Quantity values are unknown, so calculations use estimates, denoted $\boldsymbol{y}, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots$. Individual components of uncertainty are derived analytically, using complex calculus. The component of uncertainty in $\boldsymbol{y}$ due to the uncertainty of $\boldsymbol{x}_i$ is

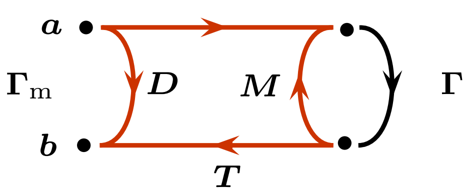
$$u_i(y) = \left| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}_i} \right| u(x_i) \ ,$$

where $u(x_i)$ is the standard uncertainty in the real (and imaginary) component of $\boldsymbol{x}_i$. $u_i(y)$ is the component of standard uncertainty in both the real and imaginary components of $\boldsymbol{y}$.[1]

2 One-port measurements

The usual one-port VNA measurement model is shown in figure 1 with the residual calibration error terms: $\boldsymbol{D}$, for directivity; $\boldsymbol{M}$, for source-match; and $\boldsymbol{T}$, for transmission-tracking. All terms are complex.

Figure 1: Residual error model of a one-port measurement of $\boldsymbol{\Gamma}$. The value reported by the VNA is $\boldsymbol{\Gamma}_{\mathrm{m}} = \boldsymbol{b}/\boldsymbol{a}$, although the quantity of interest is really $\boldsymbol{\Gamma}$.



We can express $\boldsymbol{\Gamma}$, in terms of the observed value and the residual errors, as

$$\boldsymbol{\Gamma} = \frac{\boldsymbol{\Gamma}_{\mathrm{m}} - \boldsymbol{D}}{\boldsymbol{M}(\boldsymbol{\Gamma}_{\mathrm{m}} - \boldsymbol{D}) + \boldsymbol{T}} \ . \tag{1}$$

This equation represents the usual data processing done inside a VNA.

[1] Note the uncertainties obtained are real-valued although the method is applied to a complex measurement equation.

The three complex partial derivatives of $\boldsymbol{\Gamma}$, with respect to residual directivity, source-match and tracking are

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{D}} = -\frac{\boldsymbol{T}}{[\boldsymbol{M}(\boldsymbol{\Gamma}_\mathrm{m} - \boldsymbol{D}) + \boldsymbol{T}]^2} \approx -1 \tag{2}$$

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{T}} = -\frac{\boldsymbol{\Gamma}_\mathrm{m} - \boldsymbol{D}}{[\boldsymbol{M}(\boldsymbol{\Gamma}_\mathrm{m} - \boldsymbol{D}) + \boldsymbol{T}]^2} \approx -\boldsymbol{\Gamma}_\mathrm{m} \tag{3}$$

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{M}} = -\frac{(\boldsymbol{\Gamma}_\mathrm{m} - \boldsymbol{D})^2}{[\boldsymbol{M}(\boldsymbol{\Gamma}_\mathrm{m} - \boldsymbol{D}) + \boldsymbol{T}]^2} \approx -\boldsymbol{\Gamma}_\mathrm{m}^2 \,. \tag{4}$$

The estimated residuals $\boldsymbol{D} \approx 0$, $\boldsymbol{M} \approx 0$ and $\boldsymbol{T} \approx 1$ allow us to write approximate expressions, as shown on the right. The corresponding components of uncertainty are[2]

$$u(D), \quad |\boldsymbol{\Gamma}_\mathrm{m}|\, u(T), \text{ and } \quad |\boldsymbol{\Gamma}_\mathrm{m}^2|\, u(M) \ ,$$

for directivity, reflection-tracking and source-match, respectively.

2.1 Simple flow-graph interpretation

Referring again to figure 1, flow-graph analysis can be used to describe $\boldsymbol{\Gamma}_\mathrm{m}$ by combining a few of the signal paths from $\boldsymbol{a}$ to $\boldsymbol{b}$:

1. the direct path from $\boldsymbol{a}$ to $\boldsymbol{b}$ with gain $\boldsymbol{D}$;
2. the main measurement path, through $\boldsymbol{\Gamma}$ and $\boldsymbol{T}$, with gain $\boldsymbol{T\Gamma}$;
3. a path through $\boldsymbol{\Gamma}$, twice, with one loop back through $\boldsymbol{M}$, and finally back through $\boldsymbol{T}$, with gain $\boldsymbol{TM\Gamma}^2$.

Each path represents a component of signal at $\boldsymbol{b}$, added together we obtain

$$\boldsymbol{\Gamma}_\mathrm{m} \approx \boldsymbol{D} + \boldsymbol{T}\,\boldsymbol{\Gamma} + \boldsymbol{T}\,\boldsymbol{M}\,\boldsymbol{\Gamma}^2 \,. \tag{5}$$

The complex measurement error is therefore

$$\boldsymbol{\Gamma}_\mathrm{m} - \boldsymbol{\Gamma} \approx \boldsymbol{D} + (\boldsymbol{T} - 1)\boldsymbol{\Gamma} + \boldsymbol{T}\,\boldsymbol{M}\,\boldsymbol{\Gamma}^2 \,. \tag{6}$$

Now these expressions are not easily handled by a GUM-type uncertainty analysis, which requires a function $Y = f(\cdots)$ relating a measurand to all influence quantities. In this problem, the measurand is $\boldsymbol{\Gamma}$, not $\boldsymbol{\Gamma}_\mathrm{m}$.

However, we are assuming that $\boldsymbol{D} \approx 0$, $\boldsymbol{M} \approx 0$ and $\boldsymbol{T} \approx 1$, so $\boldsymbol{\Gamma}_\mathrm{m} \approx \boldsymbol{\Gamma}$. Substituting $\boldsymbol{\Gamma}_\mathrm{m}$ for $\boldsymbol{\Gamma}$ on the right-hand side of (6) and dropping $\boldsymbol{T}$ in the last term, we obtain

$$\boldsymbol{\Gamma}_\mathrm{m} - \boldsymbol{\Gamma} \approx \boldsymbol{D} + (\boldsymbol{T} - 1)\boldsymbol{\Gamma}_\mathrm{m} + \boldsymbol{M}\,\boldsymbol{\Gamma}_\mathrm{m}^2 \,, \tag{7}$$

which can easily be differentiated with respect to the influence quantities. The sensitivities of $\boldsymbol{\Gamma}$ to $\boldsymbol{D}$, $\boldsymbol{M}$ and $\boldsymbol{T}$ that we obtain are the same as the approximate expressions in (2), (3) and (4) above.

[2] Our preferred notation uses $u(\cdot)$ for a standard uncertainty. However, because D, M and $T - 1$ are close to zero one may associate D with $u(D)$, M with $u(M)$ and $T - 1$ with $u(T)$.

2.2 Comments on the Guide's treatment

Section 6 of the Guide is concerned with complex reflection coefficient measurement using one port of a VNA. The Guide's equation-2 for these measurements is

$$\boxed{U_{\mathrm{VRC}} = D + T\Gamma + M\Gamma^2 + R_{\mathrm{VRC}}}$$

An explicit meaning is not attributed to U_{VRC}, but other terms are defined as:

Γ **:** the measured voltage reflection coefficient,

D **:** the effective directivity,

T **:** the effective tracking and non-linearity,

M **:** the effective test-port match,

R_{VRC} **:** representative of all random errors.

All quantities are real-valued magnitudes (i.e., no phase information) in linear units (as opposed to log). The magnitudes $D \approx 0$, $M \approx 0$ and $T \approx 1$ are characteristic of residual errors, after the VNA has been calibrated. The term $R_{\mathrm{VRC}} \approx 0$ is a catch-all for random errors.

It is not clear whether the Guide's equation-2 is intended to be similar to equation (5) or (7).[3] If it is (5), then the use of the letter 'U' on the left-hand side is unfortunate, as that quantity is not an error or an uncertainty.

In any case, the description of the measurement model should really be made in terms of complex quantities. The YS method of uncertainty analysis can then be applied, which allows us to work with the real-valued magnitudes.

2.3 Summary

The measurement model should be expressed in terms of complex quantities: the Guide's equation-2 is unhelpful in this regard and should be changed. It makes little difference whether measurement equation is stated, like (5)

$$\mathbf{\Gamma}_{\mathrm{m}} \approx \boldsymbol{D} + \boldsymbol{T}\,\mathbf{\Gamma} + \boldsymbol{T}\,\boldsymbol{M}\,\mathbf{\Gamma}^2 \,,$$

or an equation for the measurement *error*, like (7)

$$\mathbf{\Gamma}_{\mathrm{m}} - \mathbf{\Gamma} \approx \boldsymbol{D} + (\boldsymbol{T} - 1)\mathbf{\Gamma}_{\mathrm{m}} + \boldsymbol{T}\,\boldsymbol{M}\,\mathbf{\Gamma}_{\mathrm{m}}^2 \,.$$

In either case, a YS analysis obtains the following components of uncertainty[4]

$$D, \quad |\mathbf{\Gamma}_{\mathrm{m}}|\,T, \text{ and } \quad |\mathbf{\Gamma}_{\mathrm{m}}^2|\,M \;.$$

These components of uncertainty need to be clearly stated.

We note that it is always important to identify the quantity intended to be measured $\mathbf{\Gamma}$, as this is the subject of any GUM uncertainty analysis.

[3] The catch-all term for random measurement errors, R_{VRC}, could of course be added to (5) and (7). It does not arise in our analysis because only the systematic errors associated with calibration constants have been considered.

[4] The more formal GUM analysis of one-port VNA measurements also identifies these three components of uncertainty for directivity, tracking and mismatch, respectively, but the notation is slightly different

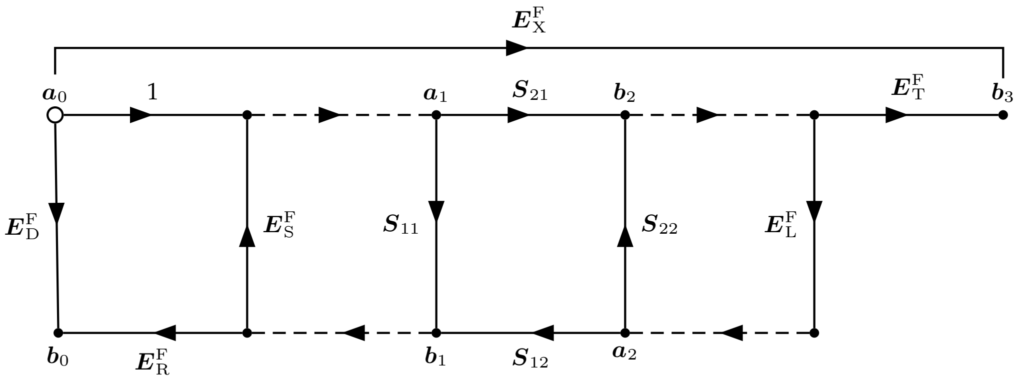
$$u(D), \quad |\mathbf{\Gamma}_{\mathrm{m}}|\,u(T), \text{ and } \quad |\mathbf{\Gamma}_{\mathrm{m}}^2|\,u(M) \;.$$

Since D, M and $T-1$ are all ≈ 0, we may associate D with $u(D)$, M with $u(M)$ and $T-1$ with $u(T)$.

3 Two-port reflection measurements

The conventional error model for two-port VNA measurements consists of twelve error terms. Figure 2 showes one half of this model dealing with the forward VNA configuration only. A different notation is used because of the model's complexity. A superscript ('F' or 'R') denotes whether a term is in the forward or reverse configuration and a subscript identifies the type of error (directivity, source-match, etc). Cross-talk terms are denoted $\boldsymbol{E}_\mathrm{X}^\mathrm{F}$ and $\boldsymbol{E}_\mathrm{X}^\mathrm{R}$, $\boldsymbol{E}_\mathrm{T}^\mathrm{F}$ and $\boldsymbol{E}_\mathrm{T}^\mathrm{R}$ are forward- and reverse-tracking errors while $\boldsymbol{E}_\mathrm{L}^\mathrm{F}$ and $\boldsymbol{E}_\mathrm{L}^\mathrm{R}$ are test-port match errors. Also, note that these terms represent *residual* VNA errors, after calibration.[5]

Figure 2: Signal flow-graph for the 'forward' configuration of the usual '12-term' VNA error model. A signal is introduced at the open-circle node on the left. The $\boldsymbol{S}_{ij}$ are complex S-parameters of a DUT connected between the measurement ports.



The conventional VNA data-processing equations that convert a set of VNA-observed S-parameters (indicated by the superscript 'M') into estimates of DUT S-parameters are:

[5]The $\boldsymbol{E}_\mathrm{x}^\mathrm{y}$ notation is more commonly used when discussing calibration methods, where terms correspond to the physical errors, not residuals. However, we may consider a second calibration to be performed on an already calibrated VNA, in which case the error terms of the second-order calibration are indeed the residual errors of the initial calibration.

$$\boldsymbol{S}_{11} = \frac{\left\{\left(\frac{\boldsymbol{S}_{11}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}}\right)\left[1+\left(\frac{\boldsymbol{S}_{22}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{R}}\right]\right\} - \left[\left(\frac{\boldsymbol{S}_{21}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{F}}}\right)\left(\frac{\boldsymbol{S}_{12}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{L}}^{\mathrm{F}}\right]}{\left[1+\left(\frac{\boldsymbol{S}_{11}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{F}}\right]\left[1+\left(\frac{\boldsymbol{S}_{22}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{R}}\right] - \left[\left(\frac{\boldsymbol{S}_{21}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{F}}}\right)\left(\frac{\boldsymbol{S}_{12}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{L}}^{\mathrm{F}}\boldsymbol{E}_{\mathrm{L}}^{\mathrm{R}}\right]} \tag{8}$$

$$\boldsymbol{S}_{21} = \frac{\left[1+\left(\frac{\boldsymbol{S}_{22}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{R}}}\right)\left(\boldsymbol{E}_{\mathrm{S}}^{\mathrm{R}}-\boldsymbol{E}_{\mathrm{L}}^{\mathrm{F}}\right)\right]\left(\frac{\boldsymbol{S}_{21}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{F}}}\right)}{\left[1+\left(\frac{\boldsymbol{S}_{11}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{F}}\right]\left[1+\left(\frac{\boldsymbol{S}_{22}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{R}}\right] - \left[\left(\frac{\boldsymbol{S}_{21}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{F}}}\right)\left(\frac{\boldsymbol{S}_{12}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{L}}^{\mathrm{F}}\boldsymbol{E}_{\mathrm{L}}^{\mathrm{R}}\right]} \tag{9}$$

$$\boldsymbol{S}_{12} = \frac{\left[1+\left(\frac{\boldsymbol{S}_{11}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}}\right)\left(\boldsymbol{E}_{\mathrm{S}}^{\mathrm{F}}-\boldsymbol{E}_{\mathrm{L}}^{\mathrm{R}}\right)\right]\left(\frac{\boldsymbol{S}_{12}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{R}}}\right)}{\left[1+\left(\frac{\boldsymbol{S}_{11}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{F}}\right]\left[1+\left(\frac{\boldsymbol{S}_{22}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{R}}\right] - \left[\left(\frac{\boldsymbol{S}_{21}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{F}}}\right)\left(\frac{\boldsymbol{S}_{12}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{L}}^{\mathrm{F}}\boldsymbol{E}_{\mathrm{L}}^{\mathrm{R}}\right]} \tag{10}$$

$$\boldsymbol{S}_{22} = \frac{\left\{\left(\frac{\boldsymbol{S}_{22}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{R}}}\right)\left[1+\left(\frac{\boldsymbol{S}_{11}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{F}}\right]\right\} - \left[\left(\frac{\boldsymbol{S}_{21}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{F}}}\right)\left(\frac{\boldsymbol{S}_{12}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{L}}^{\mathrm{R}}\right]}{\left[1+\left(\frac{\boldsymbol{S}_{11}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{F}}\right]\left[1+\left(\frac{\boldsymbol{S}_{22}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{D}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{R}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{S}}^{\mathrm{R}}\right] - \left[\left(\frac{\boldsymbol{S}_{21}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{F}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{F}}}\right)\left(\frac{\boldsymbol{S}_{12}^{\mathrm{M}}-\boldsymbol{E}_{\mathrm{X}}^{\mathrm{R}}}{\boldsymbol{E}_{\mathrm{T}}^{\mathrm{R}}}\right)\boldsymbol{E}_{\mathrm{L}}^{\mathrm{F}}\boldsymbol{E}_{\mathrm{L}}^{\mathrm{R}}\right]} \tag{11}$$

The task of obtaining the sensitivity coefficients needed for the YS method is discussed in Appendix A. The general expressions are simplified considerably by making the assumptions that $\boldsymbol{E}_{\mathrm{D}}^{x} \approx 0$, $\boldsymbol{E}_{\mathrm{S}}^{x} \approx 0$, $\boldsymbol{E}_{\mathrm{L}}^{x} \approx 0$, $\boldsymbol{E}_{\mathrm{R}}^{x} \approx 1$ and $\boldsymbol{E}_{\mathrm{T}}^{x} \approx 1$, where x can be 'F' or 'R'. We obtain the following sensitivity coefficients

$$\frac{\partial \boldsymbol{S}_{11}}{\partial \boldsymbol{E}_{\mathrm{D}}^{\mathrm{F}}} \approx -1$$
$$\frac{\partial \boldsymbol{S}_{11}}{\partial \boldsymbol{E}_{\mathrm{S}}^{\mathrm{F}}} \approx -(\boldsymbol{S}_{11}^{\mathrm{M}})^2$$
$$\frac{\partial \boldsymbol{S}_{11}}{\partial \boldsymbol{E}_{\mathrm{R}}^{\mathrm{F}}} \approx -\boldsymbol{S}_{11}^{\mathrm{M}}$$
$$\frac{\partial \boldsymbol{S}_{11}}{\partial \boldsymbol{E}_{\mathrm{L}}^{\mathrm{F}}} \approx -\boldsymbol{S}_{21}^{\mathrm{M}}\boldsymbol{S}_{12}^{\mathrm{M}}$$

The remaining eight sensitivity coefficients are all approximately zero:

The components of uncertainty for a two-port reflection measurement are therefore

$$u(E_{\mathrm{D}}^{\mathrm{F}}), \quad |\boldsymbol{S}_{11}^{\mathrm{M}}|^2 \cdot u(E_{\mathrm{S}}^{\mathrm{F}}), \quad |\boldsymbol{S}_{11}^{\mathrm{M}}| \cdot u(E_{\mathrm{R}}^{\mathrm{F}}) \quad \text{and} \quad |\boldsymbol{S}_{21}^{\mathrm{M}}\boldsymbol{S}_{12}^{\mathrm{M}}| \cdot u(E_{\mathrm{L}}^{\mathrm{F}})$$

3.1 Simple flow-graph interpretation

A simplified model of a VNA two-port refection measurement is shown in figure 3, where the notation has reverted to that used in the Guide.

An approximate expression for the measurement error in this case is

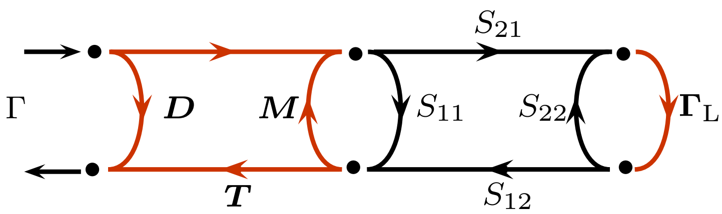
$$\boldsymbol{\Gamma}_{\mathrm{m}} - \boldsymbol{\Gamma} \approx \boldsymbol{D} + (\boldsymbol{T}-1)\boldsymbol{\Gamma} + \boldsymbol{T}\,\boldsymbol{M}\,\boldsymbol{\Gamma}^2 + \boldsymbol{T}\,\boldsymbol{S}_{12}\boldsymbol{S}_{21}\,\boldsymbol{\Gamma}_{\mathrm{L}} \tag{12}$$

We recognise the first three terms in (12) as being the same as those in (6) for the one-port case from §2.1. The remaining term is for the path, from left to right, through the DUT, reflected back by $\boldsymbol{\Gamma}_{\mathrm{L}}$ and through the DUT again.

It is convenient to make some further approximations, as in §2.1: $\boldsymbol{\Gamma}_{\mathrm{m}} \approx \boldsymbol{\Gamma}$, $\boldsymbol{S}_{21}\boldsymbol{S}_{12} \approx \boldsymbol{S}_{12}^{\mathrm{M}}\boldsymbol{S}_{21}^{\mathrm{M}}$ and $\boldsymbol{T} \approx 1$ we obtain

$$\boldsymbol{\Gamma}_{\mathrm{m}} - \boldsymbol{\Gamma} \approx \boldsymbol{D} + (\boldsymbol{T}-1)\boldsymbol{\Gamma}_{\mathrm{m}} + \boldsymbol{M}\,\boldsymbol{\Gamma}_{\mathrm{m}}^2 + \boldsymbol{S}_{12}^{\mathrm{M}}\boldsymbol{S}_{21}^{\mathrm{M}}\,\boldsymbol{\Gamma}_{\mathrm{L}}\,. \tag{13}$$

Figure 3: A simplified flowgraph of a two-port VNA reflection measurement. Red branches are associated with residual VNA errors. The measurement configuration is with the signal source on the left.



The components of uncertainty are

$$u(D)\,, \quad |\boldsymbol{\Gamma}_{\mathrm{m}}| \cdot u(T)\,, \quad |\boldsymbol{\Gamma}_{\mathrm{m}}|^2 \cdot u(M)\,, \quad \text{and} \quad |\boldsymbol{S}_{12}^{\mathrm{M}}\boldsymbol{S}_{21}^{\mathrm{M}}| \cdot u(\Gamma_{\mathrm{L}})\,,$$

which are equivalent to those obtained by the more detailed YS analysis.

3.2 Comments on the Guide's treatment

Section 7.1 of the Guide is concerned with the measurement of complex reflection S-parameters with a VNA. The Guide assumes that S_{21} and S_{12} are equal and associates the symbol Γ with either S_{11} or S_{22} (S_{11} if VNA port 1 is the signal source). The reflection coefficient of the receiving VNA port is designated Γ_{L}.

The Guide's equation-5 for the two-port reflection measurement is

$$\boxed{U_{\mathrm{VRC}} = D + T\Gamma + M\Gamma^2 + R_{\mathrm{VRC}} + S_{21}^2\Gamma_{\mathrm{L}}}$$

where D, M and T have their usual meaning and again R_{VRC} is a catch-all term for random error. All terms are in linear units.

The Guide has used a 'U' notation again, although the equation appears to describe the observed value of Γ. In any case the equation written in terms of scalar quantities is not a useful error model.

A note in § 7.1.1 of the Guide suggests that an additional term

$$2 \cdot \Gamma \cdot M \cdot \Gamma_{\mathrm{L}} \cdot S_{12}^2$$

might be added to equation-5 if the *attentuation* measured is small (e.g. 3dB).

This note requires clarification. Firstly, the measurement under discussion is a reflection, not an attenuation (transmission), so the context appears wrong. Perhaps it refers to the situation when the reflection coefficient of a small-value attenuator is measured.

Secondly, the origin of the term itself is unclear and it does not arise in the detailed GUM analysis, because it is of second order smallness. This needs to be addressed.

3.3 Summary

As in §2, a measurement model expressed in terms of complex quantities is needed, like, for example, (13)

$$\boldsymbol{\Gamma}_{\mathrm{m}} - \boldsymbol{\Gamma} \approx \boldsymbol{D} + (\boldsymbol{T} - 1)\boldsymbol{\Gamma}_{\mathrm{m}} + \boldsymbol{M}\,\boldsymbol{\Gamma}_{\mathrm{m}}^2 + \boldsymbol{S}_{12}^{\mathrm{M}}\boldsymbol{S}_{21}^{\mathrm{M}}\,\boldsymbol{\Gamma}_{\mathrm{L}}\,.$$

The GUM analysis identifies four components of uncertainty:

$$u(D)\,,\quad |\boldsymbol{\Gamma}_{\rm m}|\cdot u(T)\,,\quad |\boldsymbol{\Gamma}_{\rm m}|^2\cdot u(M)\,,\quad \text{and}\quad |\boldsymbol{S}_{12}^{\rm M}\boldsymbol{S}_{21}^{\rm M}|\cdot u(\Gamma_{\rm L})$$

With the assumption that D, M, $T-1$ and $\Gamma_{\rm L}$ are all close to zero, one may associate D with $u(D)$, M with $u(M)$, $T-1$ with $u(T)$ and $\Gamma_{\rm L}$ with $u(\Gamma_{\rm L})$. These components of uncertainty are therefore the equivalent of those in the Guide, given the additional assumptions that $\Gamma_m \approx \Gamma$ and $S_{12} \approx S_{21}$.

4 Two-port transmission measurements

We now consider a measurement of $\boldsymbol{S}_{21}$, or $\boldsymbol{S}_{12}$. As in the previous section, the partial derivatives of $\boldsymbol{S}_{21}$ with respect to the residual calibration errors are required (see Appendix A). Making the assumption that the residual errors are nearly unity, or zero (as appropriate for ideal VNA operation), we obtain four non-zero sensitivity coefficients

$$\frac{\partial \boldsymbol{S}_{21}}{\partial \boldsymbol{E}_{\rm S}^{\rm F}} \approx -\boldsymbol{S}_{11}^{\rm M}\cdot\boldsymbol{S}_{21}^{\rm M}$$
$$\frac{\partial \boldsymbol{S}_{21}}{\partial \boldsymbol{E}_{\rm L}^{\rm F}} \approx -\boldsymbol{S}_{22}^{\rm M}\cdot\boldsymbol{S}_{21}^{\rm M}$$
$$\frac{\partial \boldsymbol{S}_{21}}{\partial \boldsymbol{E}_{\rm T}^{\rm F}} \approx -\boldsymbol{S}_{21}^{\rm M}$$
$$\frac{\partial \boldsymbol{S}_{21}}{\partial \boldsymbol{E}_{\rm X}^{\rm F}} = -1\,.$$

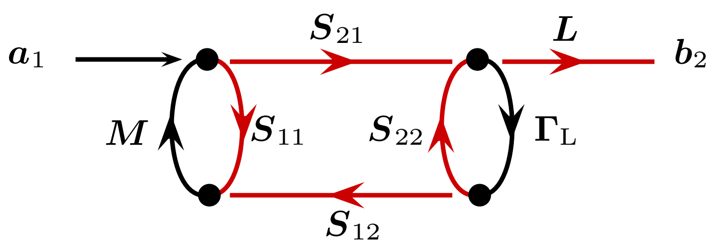
In the transmission case, it is convenient to use *relative* components of uncertainty, so we divide by $\boldsymbol{S}_{21}^{\rm M}$ and obtain the following four relative components of uncertainty

$$|\boldsymbol{S}_{11}^{\rm M}|\cdot u(E_{\rm S}^{\rm F}),\quad |\boldsymbol{S}_{22}^{\rm M}|\cdot u(E_{\rm L}^{\rm F}),\quad u(E_{\rm T}^{\rm F}),\quad \text{and}\quad u(E_{\rm X}^{\rm F})/|\boldsymbol{S}_{21}^{\rm M}|$$

4.1 Simple flow-graph interpretation

A signal flow-graph relating to the transmission measurement is shown in figure 4. Three residual calibration errors are shown: $\boldsymbol{M}$, for the source-match, $\boldsymbol{\Gamma}_{\rm L}$, for the load-match and $\boldsymbol{L}$ for the transmission-tracking (linearity, etc). A leakage path would be a parallel path from $\boldsymbol{a}$ to $\boldsymbol{b}$, but this is not shown.

Figure 4: A simplified signal flow-graph of the two-port measurement problem. The two VNA test-port match error terms are shown as well as the transmission tracking.



The transmission of this network is

$$\begin{aligned}\frac{\boldsymbol{b}_2}{\boldsymbol{a}_1} &= \frac{\boldsymbol{L}\boldsymbol{S}_{21}}{(1-\boldsymbol{M}\boldsymbol{S}_{11})(1-\boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\mathrm{L}}) - \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{12}\boldsymbol{S}_{21}} \\ &= \frac{\boldsymbol{L}\boldsymbol{S}_{21}}{1-\boldsymbol{M}\boldsymbol{S}_{11} - \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22} - \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{12}\boldsymbol{S}_{21} + \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{11}S_{22}} \ .\end{aligned}$$

Dividing both sides by $\boldsymbol{S}_{21}$ and recognising that all terms in the denominator are small compared to unity, we may write an expression for the relative measurement *error*

$$\frac{1}{\boldsymbol{S}_{21}}\frac{\boldsymbol{b}_2}{\boldsymbol{a}_1} \approx \boldsymbol{L} + \boldsymbol{M}\boldsymbol{S}_{11} + \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22} + \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{12}\boldsymbol{S}_{21} - \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{11}S_{22} \ . \tag{14}$$

To this expression we add an additional term for the leakage path, denoted here in linear units as $\boldsymbol{X}$, obtaining

$$\frac{1}{\boldsymbol{S}_{21}}\frac{\boldsymbol{b}_2}{\boldsymbol{a}_1} \approx \boldsymbol{L} + \boldsymbol{M}\boldsymbol{S}_{11} + \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22} + \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{12}\boldsymbol{S}_{21} - \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{11}S_{22} + \boldsymbol{X}/\boldsymbol{S}_{21} \ . \tag{15}$$

If we ignore the fourth and fifth terms in this equation, which involve products of the residual errors and are therefore of second-order smallness, the components of relative uncertainty are

$$u(L), \quad |\boldsymbol{S}_{11}| \cdot u(M), \quad |\boldsymbol{S}_{22}| \cdot u(L), \quad \text{and} \quad |1/\boldsymbol{S}_{21}| \cdot u(\boldsymbol{X}) \ .$$

These are equivalent to the components obtained by GUM analysis.

4.1.1 Terms in dB

The Guide only treats this problem in logarithmic units and identifies 'mismatch', 'isolation' and 'linearity' in its analysis.

In (15) we may associate $(\boldsymbol{M}\boldsymbol{S}_{11} + \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22} + \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{12}\boldsymbol{S}_{21} - \boldsymbol{M}\boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{11}S_{22})$ with mismatch, $\boldsymbol{X}/\boldsymbol{S}_{21}$ with isolation and $\boldsymbol{L}$ with 'linearity'.

Dropping the second-order mismatch terms in (15) and noting that $\boldsymbol{L} \approx 1$, while all other terms are ≈ 0, we write

$$\frac{1}{\boldsymbol{S}_{21}}\frac{\boldsymbol{b}_2}{\boldsymbol{a}_1} \approx \boldsymbol{L}\left(1 + \boldsymbol{M}\boldsymbol{S}_{11} + \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22}\right)\left(1 + \boldsymbol{X}/\boldsymbol{S}_{21}\right) \ ,$$

which can be converted to logarithmic units by making the usual approximations[6]

$$(\text{relative transmission error})_{\mathrm{dB}} \approx L_{\mathrm{dB}} + M_{\mathrm{dB}} + X_{\mathrm{dB}} \ , \tag{16}$$

where L_{dB} is associated with linearity, M_{dB} is associated with the sum $1 + \boldsymbol{M}\boldsymbol{S}_{11} + \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22}$, and X_{dB} is associated with $1 + \boldsymbol{X}/\boldsymbol{S}_{21}$.

All terms in (16) are ideally zero, so the uncertainty components (in dB) correspond to the value of these terms (i.e. the uncertainty in linearity is the departure of L_{dB} from 0, etc).

[6] Terms inside the parentheses are considered small compared to unity allowing us to use the approximations $|1+\boldsymbol{\epsilon}|^2 \approx 1 + 2\mathrm{Re}\{\boldsymbol{\epsilon}\}$ and $10\log_{10}(1+x) \approx \frac{10}{\log_e 10}\, x \quad (x \ll 1)$

4.2 Comments on the Guide's treatment

Content: In § 7.2, the Guide's equation-6 is given as an error model for transmission measurements

$$\boxed{U_{\mathrm{TM}} = L + M_{\mathrm{TM}} + I + R_{\mathrm{dB}}}$$

with terms, all in logarithmic units, defined in the following way

L : is the measured System Deviation from linearity

M_{TM} : is the calculated error term due to Mismatch

I : is the estimated or measured Cross-Talk; dA

R_{dB} : represents all the random Contributions

Mismatch is treated further in § 7.3.2, where it states that there will be an **uncertainty due to mismatch** (Guide's emphasis) given by

$$M_{\mathrm{TM}} = 20\log_{10}\frac{1 + (|MS_{11}| + |\Gamma_{\mathrm{L}}S_{22}| + |M\Gamma_{\mathrm{L}}S_{11}S_{22}| + |M\Gamma_{\mathrm{L}}S_{12}S_{21}|)}{1 - |M||\Gamma_{\mathrm{L}}|} \tag{17}$$

where the terms, in linear units now, are defined as

M : is the Effective Test Port Match

Γ_{L} : is the Effective Load Match

S_{11}, S_{12}, S_{21}, S_{22} : are the Scattering coefficients of the device being measured

Isolation (cross-talk) is treated in § 7.3.3, where the following formula is given

$$dA = \pm 20\log_{10}\left[1 + 10^{-\frac{(I-A)}{20}}\right]$$

where A is the attenuation and I is the cross-talk (both in dB).

Comments: This is a more complicated treatment than the other measurement situations, because several different expressions are needed to describe the contributions due mismatch and isolation. There is also the complication of changing between logarithmic and linear units.

A mistake has been made in the treatment of mismatch: equation (17) is incorrect for the VNA measurement problem. The equation given is appropriate when the ratio of a pair of transmission measurements is calculated. When one measurement is a 'thru', the denominator accounts for the mismatch during that measurement. However, having calibrated a VNA, separate 'thru' measurements are not usually made. Furthermore, the two second-order terms in the numerator of (17) do not arise in the GUM analysis.[7]

The Guide's expression for mismatch can therefore be be written more simply as

$$M_{\mathrm{TM}} = 20\log_{10}(1 - \boldsymbol{M}\boldsymbol{S}_{11} - \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22}) \ .$$

The Guide's reference [3] does not seem to be needed either.

[7]We note the examples given in the Guide's appendices do not consider second-order mismatch terms anyway.

4.3 Summary

A linear error model, from which uncertainty components can be obtained in both linear and logarithmic units, is useful. For instance, the relative error in a transmission measurement in linear units is

$$\frac{1}{\boldsymbol{S}_{21}}\frac{\boldsymbol{b}_2}{\boldsymbol{a}_1} \approx \boldsymbol{L} + \boldsymbol{M}\boldsymbol{S}_{11} + \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22} + \boldsymbol{X}/\boldsymbol{S}_{21} \ ,$$

or, in logarithmic units,

$$(\text{relative transmission error})_{\mathrm{dB}} \approx L_{\mathrm{dB}} + M_{\mathrm{dB}} + X_{\mathrm{dB}} \ ,$$

where $L_{\mathrm{dB}} = \log_{10}|\boldsymbol{L}|$, $M_{\mathrm{dB}} = \log_{10}|1 + \boldsymbol{M}\boldsymbol{S}_{11} + \boldsymbol{\Gamma}_{\mathrm{L}}\boldsymbol{S}_{22}|$ and $X_{\mathrm{dB}} = \log_{10}|1 + \boldsymbol{X}/\boldsymbol{S}_{21}|$.

The relative uncertainty components are, in linear units,

$$u(L), \quad |\boldsymbol{S}_{11}| \cdot u(M), \quad |\boldsymbol{S}_{22}| \cdot u(L), \quad \text{and} \quad |1/\boldsymbol{S}_{21}| \cdot u(\boldsymbol{X}) \ .$$

In logarithmic units the uncertainty components are simply

$$L_{\mathrm{dB}}, \quad M_{\mathrm{dB}} \quad \text{and} \quad X_{\mathrm{dB}} \ .$$

5 Uncertainty in the complex plane

The Guide does not advise on the evaluation of uncertainty in phase for VNA measurements. It mentions this in §7.4, with regard to transmission measurements, but it does not comment on phase uncertainty for reflection measurements either. This is unfortunate as there is sufficient information to allow uncertainty statements to be made about complex S-parameters, both for reflection and transmission measurements.

The YS method of uncertainty calculation is compatible with the recommendations of the Guide. Standard uncertainties for the real and imaginary components of a result can be calculated. The YS method forces us to assume that there are equal and independent uncertainties associated with the real and imaginary components of a complex quantity. This means that the uncertainty of a measured complex quantity can be represented by a circle in the complex plane, as shown in figure 5. The radius should be scaled by a two-dimensional coverage factor $k_{2,p}$, where $100p\%$ is the level of confidence.[8]

5.1 Uncertainty in phase

There is often interest in obtaining uncertainty statements in polar coordinates. In that case, if the magnitude of the measured value is

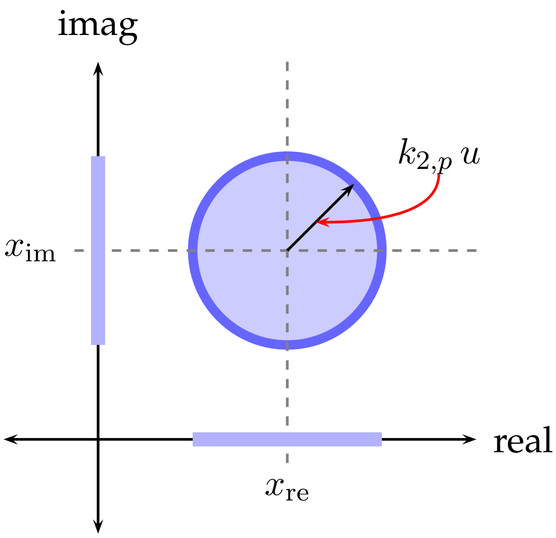
$$|\boldsymbol{x}| = \sqrt{x_{\mathrm{re}}^2 + x_{\mathrm{im}}^2} \ ,$$

and the the standard uncertainty of the magnitude is

$$u(|\boldsymbol{x}|) = u \ ,$$

[8]Note that coverage factors in two dimensions are different from those in one dimension. A two-dimensional uncertainty region should capture *both* the real and imaginary components. A one-dimensional uncertainty interval should capture only one component.

Figure 5: A region in the complex plane representing the uncertainty of a measured value $\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}\, x_{\mathrm{im}}$. The standard uncertainty in the real and imaginary components is u. The coverage factor $k_{2,p}$ determines extent of the region, according to the required level of confidence.



where

$$u = u(x_{\mathrm{re}}) = u(x_{\mathrm{im}}) \ ,$$

the standard uncertainty in the real and imaginary components.

A simple geometrical transformation is sometimes adequate to obtain a standard uncertainty in phase. This is shown in figure 6. The angle subtended by a circle with a radius equal to the standard uncertainty represents the uncertainty in phase, written as

$$u(\phi) \approx \tan^{-1} \frac{u}{|\boldsymbol{x}|} \ .$$

It should be clear that this transformation is only reliable when the angle subtended is not too large.[9]

5.2 Coverage factors

Consideration should be given to the coverage factor when reporting an expanded uncertainty.

When the quantity intended to be measured is complex-valued, a region in the complex plane around the measurement result is of interest. To set the size of this region correctly, a two dimensional coverage factor should be used

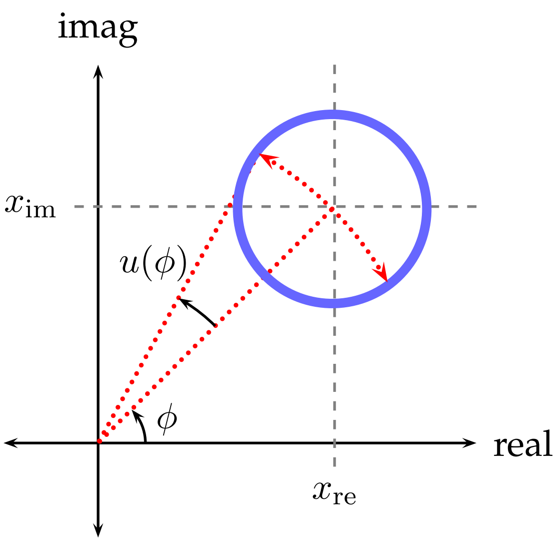
$$k_{2,p}^2 = \frac{2\nu}{\nu - 1} F_{2,\nu-1}(p) \ ,$$

where ν is the number of effective degrees of freedom and F is the F-distribution (when $\nu = \infty$, $k_{2,p}^2 = \chi_{2,p}^2$).[10]

[9] A method of checking the validity of the uncertainty after transformation has been described in [4, Appendix A].

[10] The 'total-variance' method for calculating a number of effective degrees-of-freedom for a complex quantity can be used to obtain ν [5, Appendix 1].

Figure 6: Construction used to obtain an uncertainty in phase. The radius of the circle is the standard uncertainty u.



When the quantity intended to be measured is real (e.g., if *only* one or other of the magnitude or phase is of interest), an interval around the measured value is required. In that case, the usual procedures of the GUM should be followed, including the conventional coverage factor and effective degrees-of-freedom calculation (Welch-Satterthwaite equation) [2, Appendix G].

6 Conclusions

This report has reviewed the relationship between common error models for one-port and two-port VNA measurements and the equations given as error models in the EURAMET Guide for evaluating the uncertainty in such measurements.

A formal analysis has verified the principal components of uncertainty in each type of measurement cited in the Guide. However, some of the lesser uncertainty terms are not reproduced in the analysis.

The Guide's use of error models could be clarified and simplified. Models are incorrectly presented as real-valued quantity equations and, in several cases, the intended subject of the equation is actually unclear. From a practical viewpoint, these equations serve only to suggest to the reader the terms that the Guide considers to be significant contributors to the uncertainty. They cannot be used to analyse the measurement problem. In the case of transmission measurements, an equation for mismatch seems to be incorrect. The Guide could be improved by clearly stating simple measurement error models, in terms of complex quantities, and by explicitly stating the principal components of uncertainty associated with each of the models.

The Guide could easily be extended to make statements about the uncertainty of complex quantities and, in some cases, about the phase uncertainty of results. Since it was drafted, our understanding about complex uncertainties has improved and it would now be possible to extend the Guide with some additional advice in this area.

7 Addendum

After preparing this report, we became aware of two papers by Stenarson and Yhland that offer interesting insight in the context of the EURAMET Guide and the assessment of VNA residual errors [6] [7].

Acknowledgement

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A Two-port sensitivity analysis

The analysis of equations (8) and (9) to obtain sensitivity coefficients is laborious. An automatic symbolic mathematics tool was used to do this, and to apply the error estimates that led to the approximate expressions in the body of this report.

This appendix documents the expressions obtained by the software before simplification. The results are given only in the raw format generated by the software.

In the what follows, some intermediate variables have been defined to simplify the expressions. In particular, the denominator in (8) and (9) is referred to as

$$N = \left[1 + \left(\frac{S_{11}^{\mathrm{M}} - E_{\mathrm{D}}^{\mathrm{F}}}{E_{\mathrm{R}}^{\mathrm{F}}}\right) E_{\mathrm{S}}^{\mathrm{F}}\right] \left[1 + \left(\frac{S_{22}^{\mathrm{M}} - E_{\mathrm{D}}^{\mathrm{R}}}{E_{\mathrm{R}}^{\mathrm{R}}}\right) E_{\mathrm{S}}^{\mathrm{R}}\right] - \left[\left(\frac{S_{21}^{\mathrm{M}} - E_{\mathrm{X}}^{\mathrm{F}}}{E_{\mathrm{T}}^{\mathrm{F}}}\right) \left(\frac{S_{12}^{\mathrm{M}} - E_{\mathrm{X}}^{\mathrm{R}}}{E_{\mathrm{T}}^{\mathrm{R}}}\right) E_{\mathrm{L}}^{\mathrm{F}} E_{\mathrm{L}}^{\mathrm{R}}\right]$$

and the four simple difference terms with the measured S-parameters are defined as

$$\begin{aligned} m_{11} &= S_{11}^{\mathrm{M}} - E_{\mathrm{D}}^{\mathrm{F}} \\ m_{12} &= S_{12}^{\mathrm{M}} - E_{\mathrm{X}}^{\mathrm{R}} \\ m_{21} &= S_{21}^{\mathrm{M}} - E_{\mathrm{X}}^{\mathrm{F}} \\ m_{22} &= S_{22}^{\mathrm{M}} - E_{\mathrm{D}}^{\mathrm{R}} \end{aligned}$$

A.1 Sensitivity of S_{11}

```
d(s11)_d(E_DF) = (E_SF*(E_RR + E_SR*m_22)*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22) -
E_LF*E_RF*E_RR*m_12*m_21) + E_RF*E_RR*E_TF*E_TR*N*(-E_RR - E_SR*m_22))
/(E_RF**2*E_RR**2*E_TF*E_TR*N**2)

d(s11)_d(E_LF) = (E_LR*m_12*m_21*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) - E_RF*E_RR*E_TF*E_TR*N*m_12*m_21)
/(E_RF*E_RR*E_TF**2*E_TR**2*N**2)

d(s11)_d(E_RF) = (E_SF*m_11*(E_RR + E_SR*m_22)*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) - E_RF*E_RR*E_TF*E_TR*N*m_11*(E_RR + E_SR*m_22))
/(E_RF**3*E_RR**2*E_TF*E_TR*N**2)

d(s11)_d(E_SF) = -m_11*(E_RR + E_SR*m_22)*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21)
/(E_RF**2*E_RR**2*E_TF*E_TR*N**2)

d(s11)_d(E_TF) = (-E_LF*E_LR*m_12*m_21*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) + E_LF*E_RF*E_RR*E_TF*E_TR*N*m_12*m_21)
/(E_RF*E_RR*E_TF**3*E_TR**2*N**2)

d(s11)_d(E_XF) = (-E_LF*E_LR*m_12*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) + E_LF*E_RF*E_RR*E_TF*E_TR*N*m_12)
/(E_RF*E_RR*E_TF**2*E_TR**2*N**2)

d(s11)_d(E_DR) = (E_SR*(E_RF + E_SF*m_11)*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) - E_RF*E_RR*E_SR*E_TF*E_TR*N*m_11)
/(E_RF**2*E_RR**2*E_TF*E_TR*N**2)

d(s11)_d(E_LR) = E_LF*m_12*m_21*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21)
/(E_RF*E_RR*E_TF**2*E_TR**2*N**2)
```

```
d(s11)_d(E_RR) = (E_SR*m_22*(E_RF + E_SF*m_11)*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) - E_RF*E_RR*E_SR*E_TF*E_TR*N*m_11*m_22)
/(E_RF**2*E_RR**3*E_TF*E_TR*N**2)

d(s11)_d(E_SR) = (-m_22*(E_RF + E_SF*m_11)*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) + E_RF*E_RR*E_TF*E_TR*N*m_11*m_22)
/(E_RF**2*E_RR**2*E_TF*E_TR*N**2)

d(s11)_d(E_TR) = (-E_LF*E_LR*m_12*m_21*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) + E_LF*E_RF*E_RR*E_TF*E_TR*N*m_12*m_21)
/(E_RF*E_RR*E_TF**2*E_TR**3*N**2)

d(s11)_d(E_XR) = (-E_LF*E_LR*m_21*(E_TF*E_TR*m_11*(E_RR + E_SR*m_22)
- E_LF*E_RF*E_RR*m_12*m_21) + E_LF*E_RF*E_RR*E_TF*E_TR*N*m_21)
/(E_RF*E_RR*E_TF**2*E_TR**2*N**2)
```

A.2 Sensitivity of S_{21}

```
d(S21)_d(E_DF) = E_SF*m_21*(E_RR + E_SR*m_22)*(E_RR + m_22*(E_SR - E_LF))
/(E_RF*E_RR**2*E_TF*N**2)

d(S21)_d(E_LF) = (E_LR*m_12*m_21**2*(E_RR + m_22*(E_SR - E_LF)) - E_TF*E_TR*N*m_21*m_22)
/(E_RR*E_TF**2*E_TR*N**2)

d(S21)_d(E_RF) = E_SF*m_11*m_21*(E_RR + E_SR*m_22)*(E_RR + m_22*(E_SR - E_LF))
/(E_RF**2*E_RR**2*E_TF*N**2)

d(S21)_d(E_SF) = -m_11*m_21*(E_RR + E_SR*m_22)*(E_RR + m_22*(E_SR - E_LF))
/(E_RF*E_RR**2*E_TF*N**2)

d(S21)_d(E_TF) = (-E_LF*E_LR*m_12*m_21**2*(E_RR + m_22*(E_SR - E_LF))
- E_TF*E_TR*N*m_21*(E_RR + m_22*(E_SR - E_LF)))
/(E_RR*E_TF**3*E_TR*N**2)

d(S21)_d(E_XF) = (E_TF*E_TR*N*(-E_RR - m_22*(E_SR - E_LF))
- E_LF*E_LR*m_12*m_21*(E_RR + m_22*(E_SR - E_LF)))
/(E_RR*E_TF**2*E_TR*N**2)

d(S21)_d(E_DR) = (E_SR*m_21*(E_RF + E_SF*m_11)*(E_RR + m_22*(E_SR - E_LF))
- E_RF*E_RR*N*m_21*(E_SR - E_LF))
/(E_RF*E_RR**2*E_TF*N**2)

d(S21)_d(E_LR) = E_LF*m_12*m_21**2*(E_RR + m_22*(E_SR - E_LF))
/(E_RR*E_TF**2*E_TR*N**2)

d(S21)_d(E_RR) = (E_SR*m_21*m_22*(E_RF + E_SF*m_11)*(E_RR + m_22*(E_SR - E_LF))
- E_RF*E_RR*N*m_21*m_22*(E_SR - E_LF))
/(E_RF*E_RR**3*E_TF*N**2)

d(S21)_d(E_SR) = (-m_21*m_22*(E_RF + E_SF*m_11)*(E_RR + m_22*(E_SR - E_LF))
+ E_RF*E_RR*N*m_21*m_22)
/(E_RF*E_RR**2*E_TF*N**2)

d(S21)_d(E_TR) = -E_LF*E_LR*m_12*m_21**2*(E_RR + m_22*(E_SR - E_LF))
/(E_RR*E_TF**2*E_TR**2*N**2)

d(S21)_d(E_XR) = -E_LF*E_LR*m_21**2*(E_RR + m_22*(E_SR - E_LF))
/(E_RR*E_TF**2*E_TR*N**2)
```

A.3 Analysis by Stumper

A sensitivity analysis of 2-port VNA measurements has been published by Stumper (although he did not consider leakage errors) [8]. The sensitivity coefficients for S_{11} can be obtained directly from Stumper's equation-6, and those for S_{21} indirectly from his equation-7.

Although the two formulations should be equivalent, those of Stumper involve fewer terms, because they are expressed in terms of the device S-parameters rather than measured S-parameters.

We use the measured S-parameters in this report, because that information is available (the device parameters are never exactly known). So we obtain sensitivities expressed in terms of known quantities.

This choice is of no practical importance, however. Had Stumper's expressions been used, an additional step of approximating the device S-parameters with measured ones would have been required and the sign of the sensitivity coefficients would have changed. This would not have altered the components of uncertainty finally obtained.