Some considerations related to the evaluation of measurement uncertainty for complex-valued quantities in radio frequency measurements

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Abstract

Several issues are discussed that relate to the evaluation of measurement uncertainty for complex-valued quantities in radio frequency measurements. In situations where there is information about the magnitude of a complex quantity, but not phase, uniform distributions in the form of a disk or a ring may be appropriate representations for the uncertainty. Variance-covariance matrices for these distributions are given for use in bivariate uncertainty calculations. The situation in which an uncertainty statement is provided in polar coordinates is also discussed. Such uncertainty statements need to be transformed into the real-imaginary coordinate system for the preferred method of uncertainty calculation. A simple transformation procedure is described together with a method to assess its accuracy.

1 Introduction

In radio frequency and microwave (RF) measurements many quantities of interest are complex valued. The *Guide to the Expression of Uncertainty in Measurement* (*Guide*) does not accommodate the case of complex-valued quantities [1].[1] However, the *Law of Propagation of Uncertainty* (LPU) for real-valued quantities presented in the *Guide* is a univariate form of a more general multivariate procedure [2]. Based on this, extensions of the LPU have been reported for complex uncertainty calculations [3, 4]. Moreover, the LPU can be applied to complex uncertainty calculations if the real and imaginary components are treated as a distinct measurands and the correlation between them, arising from shared influence quantities, is calculated (see [1, H.9]). This is equivalent to the bivariate methods for propagating uncertainty described in [3,4], which use complex numbers and matrix formulations.

In all these uncertainty calculation procedures, the quantities propagated are the variances and covariances of the probability distributions involved. In the complex-valued case, a 2-by-2 variance-covariance matrix is associated with the uncertainty of an estimate of an influence quantity, or of an estimate of the measurand.

In general, a measurement uncertainty budget itemizes the sources of uncertainty in a measurement and quantifies their combined influence on the result. Often, uncertainty contributions can be characterised using knowledge about the physical measurement system, rather than information gathered by statistical techniques. Such contributions

[1] There is an on-going effort to revise and expand the scope of the *Guide*, coordinated by the Joint Committee for Guides in Metrology. See `http://www.bipm.org/en/committees/jc/jcgm/`.

are termed 'type-B' in the vernacular of the *Guide*. For example, the uncertainty inherent in the finite resolution of a digital instrument is often associated with a uniform probability distribution.

This communication shows how to obtain variance-covariance information for some common situations that arise in RF measurements. The next section presents several type-B uncertainties that represent a total lack of information about phase. These are associated with uniform density distributions in the form of a disk or a ring in the complex plane. Section 3 then describes a way to convert from uncertainty statements given in polar coordinates to variance-covariance matrices in rectangular (real-imaginary) coordinates.

2 Uniform disk and ring distributions

In RF measurements the magnitude of a quantity is sometimes known but no information about the phase is available. In such cases, a bivariate probability distribution of uniform density and circular symmetry is usually an appropriate choice for the type-B uncertainty. For instance,

- a uniform ring distribution may represent the uncertainty associated with a quantity where the magnitude is known, but phase is not. If the ring radius a is equal to the magnitude, the covariance matrix is

$$\frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}. \tag{1}$$

- A uniform disk distribution may be used to represent the uncertainty when there is an upper limit on the magnitude, but no information about phase. If the disk radius a is equal to the magnitude limit, the covariance matrix is

$$\frac{1}{4}\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}. \tag{2}$$

These are limiting cases of an annular distribution of uniform density, with internal radius b and external radius a. The ring and disk cases correspond to: $b \to a$ and $b \to 0$, respectively. The covariance matrix of an annulus is

$$\frac{1}{4}\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}.$$

These covariance matrices for uniform bivariate probability distributions are analogous to the moment of inertia tensors of solid bodies in classical mechanics. They can be found by considering the moments of inertia, about the x and y axes, of a uniform disk, or ring, of unit mass lying in the xy plane. Such results are readily available (see, e.g., [5]).

It is interesting to note that the ring distribution is related to the univariate arcsine distribution mentioned in the *Guide*. A ring distribution centered on the origin with

radius a, when projected onto the real or imaginary axis,[2] gives rise to an arcsine distribution with limits $\pm a$ and a standard deviation $a/\sqrt{2}$ [6]. An arcsine uncertainty can be used, for example, when there is cyclic variability in a quantity between finite limits. The close relationship between the bivariate ring distribution and the univariate arcsine distribution will be illustrated below, in the example of section 2.2.

2.1 Notation

Often in metrology, the distinction between between a quantity and its estimate is not made explicit by notation, however, in this section, a notation is used to avoid potential confusion. Suppose, for example, that the magnitude $|\mathbf{\Gamma}|$ of a complex quantity $\mathbf{\Gamma}$ is known but the phase is not (bold font indicates a complex-valued quantity). We might say that an 'estimate' of $\mathbf{\Gamma}$ is zero and that a ring distribution is associated with the uncertainty (of radius $|\mathbf{\Gamma}|$). However, to write $\mathbf{\Gamma} = 0$, meaning the estimate, is confusing because it also suggests that ring radius $|\mathbf{\Gamma}| = 0$, which is not the case. To make the distinction clear in the remainder of this section, we write estimates with a caret, as in $\widehat{\mathbf{\Gamma}} = 0$.

2.2 Example: mismatch

There is an RF measurement scenario that is often discussed in relation to the problem of lack of phase information. The situation is as follows. An RF signal generator and power sensor are directly connected. The non-zero reflection coefficients of the generator and sensor, $\mathbf{\Gamma}_\mathrm{g}$ and $\mathbf{\Gamma}_\mathrm{s}$ respectively, give rise to a standing wave pattern.

In one analysis of this problem, the signal amplitude incident at the sensor is

$$\mathbf{b} = \frac{\mathbf{b}_\mathrm{g}}{1 - \mathbf{\Gamma}_\mathrm{g}\mathbf{\Gamma}_\mathrm{s}} \ , \tag{3}$$

where $\mathbf{b}_\mathrm{g}$ is the signal amplitude that would be delivered to an ideal sensor, with $\mathbf{\Gamma}_\mathrm{s} = 0$.

Often, some information will be available about the magnitudes $|\mathbf{\Gamma}_\mathrm{g}|$ and $|\mathbf{\Gamma}_\mathrm{s}|$, which are small, but nothing will be known about the phases. So, the estimates $\widehat{\mathbf{\Gamma}}_\mathrm{g}$ and $\widehat{\mathbf{\Gamma}}_\mathrm{s}$ are taken as zero and a ring, or disk, distribution used as an appropriate type-B uncertainty.

There is a problem with propagating uncertainty in equation (3) when $\widehat{\mathbf{\Gamma}}_\mathrm{g} = 0$ and $\widehat{\mathbf{\Gamma}}_\mathrm{s} = 0$. The first-order partial derivatives of the denominator, $1 - \mathbf{\Gamma}_\mathrm{g}\mathbf{\Gamma}_\mathrm{s}$, with respect to $\mathbf{\Gamma}_\mathrm{g}$ and $\mathbf{\Gamma}_\mathrm{s}$ evaluate to zero. So there will be no contribution from the denominator to the uncertainty of an estimate of $\mathbf{b}$. This can be circumvented if we treat the product as a single term

$$\mathbf{\Gamma} = \mathbf{\Gamma}_\mathrm{g}\mathbf{\Gamma}_\mathrm{s} \ , \tag{4}$$

with an estimated value of zero and an appropriate uncertainty.[3] The complete lack of phase information for $\mathbf{\Gamma}_\mathrm{g}$ and $\mathbf{\Gamma}_\mathrm{s}$ can be retained in $\mathbf{\Gamma}$.

Two cases are important to consider

1. Magnitudes $|\mathbf{\Gamma}_\mathrm{g}|$ and $|\mathbf{\Gamma}_\mathrm{s}|$ are known, so a ring distribution of radius $\Gamma^\mathrm{ring} = |\mathbf{\Gamma}|$ is appropriate.

[2] In other words, the marginal distribution.

[3] A more general discussion about the problems associated with propagating uncertainty when a measurement function is non-linear is given in [7].

2. An upper bound $\Gamma_{\mathrm{g}}^{\max}$ for $|\mathbf{\Gamma}_{\mathrm{g}}|$ is known and $|\mathbf{\Gamma}_{\mathrm{s}}|$ is known, so a disk distribution of radius $\Gamma^{\mathrm{disk}} = \Gamma_{\mathrm{g}}^{\max} |\mathbf{\Gamma}_{\mathrm{s}}|$ can be used.

The measurement scenario is usually presented in the context of a power measurement, in which case the quantity of interest is the amount of power dissipated in the sensor. Further analysis gives this quantity as

$$P_{\mathrm{s}} = P_{\mathrm{g}} \frac{1 - |\mathbf{\Gamma}_{\mathrm{s}}|^2}{|1 - \mathbf{\Gamma}|^2} \,, \tag{5}$$

where P_{g} is the power delivered when $\mathbf{\Gamma}_{\mathrm{s}} = 0$. The denominator,

$$M = |1 - \mathbf{\Gamma}|^2 \,, \tag{6}$$

is sometimes called the *mismatch* error. We now consider this term for cases 1 and 2 above, choosing $\widehat{\mathbf{\Gamma}} = 0$ and a ring, or disk, distribution to represent the uncertainty. The appropriate covariance matrices, $\mathbf{V}_\Gamma$, are given by (1), or (2), respectively, with the appropriate radius values.

Following the procedure for bivariate uncertainty propagation described in [3] or [4], we can obtain the covariance matrix associated with the mismatch

$$\mathbf{V}_M = \mathbf{J}_M \, \mathbf{V}_\Gamma \, \mathbf{J}'_M \,, \tag{7}$$

where $\mathbf{J}_M$, the Jacobian matrix of M, is

$$\mathbf{J}_M = \begin{bmatrix} 2(\widehat{\Gamma}_{\mathrm{re}} - 1) & 2\widehat{\Gamma}_{\mathrm{im}} \\ 0 & 0 \end{bmatrix}$$

and $\widehat{\Gamma}_{\mathrm{re}}$ and $\widehat{\Gamma}_{\mathrm{im}}$ are the real and imaginary components of $\widehat{\mathbf{\Gamma}}$. Since $\widehat{\mathbf{\Gamma}} = 0$, we obtain in case 1, when the uncertainty is a ring,

$$\mathbf{V}_M = \begin{bmatrix} 2\left(\Gamma^{\mathrm{ring}}\right)^2 & 0 \\ 0 & 0 \end{bmatrix}$$

and in case 2, when the uncertainty is a disk,

$$\mathbf{V}_M = \begin{bmatrix} \left(\Gamma^{\mathrm{disk}}\right)^2 & 0 \\ 0 & 0 \end{bmatrix} .$$

The mismatch error is a real-valued quantity, so the first diagonal term in these matrices is the associated standard variance.

The result for case 1 could also have been obtained using the univariate methods of the *Guide*. A lack of phase information for $\mathbf{\Gamma}$ can be thought of as a cyclic variation in the quantity

$$x = |1 - \mathbf{\Gamma}| \,.$$

An arcsine distribution with limits $1 \pm |\mathbf{\Gamma}|$ can be associated with the uncertainty of $\widehat{x} = 1$, assuming $\widehat{\mathbf{\Gamma}} = 0$. So the standard uncertainty $u(x) = |\mathbf{\Gamma}|/\sqrt{2}$ and the uncertainty in the mismatch, $u(M) = \sqrt{2}|\mathbf{\Gamma}|$, can then be found by applying the LPU to (6) [1, Ch. 5].

In practice, the situation in case 2 will be encountered, because the reflection coefficient of a generator is difficult to measure and an upper bound on the magnitude may be easier to estimate. We see that the uncertainty associated with case 2 is a factor of $\sqrt{2}$ lower than case 1, when the radii of the associated distributions are equal. So this result, which is not well known, may be useful.

3 Converting uncertainty between polar and rectangular coordinates

In many cases it is 'natural' to describe the behaviour of RF components in the polar coordinate system. For example, an attenuator is designed primarily to reduce signal magnitude and a section of transmission line can be used to introduce a phase shift, with minimal change to amplitude. Nevertheless, the non-linearity inherent in the transformation between the polar and rectangular coordinate systems does not preserve functions of statistical distributions, like the mean and variance. For this reason, serious computational errors can arise if data are processed inappropriately. Use of the real-imaginary rectangular coordinate system is strongly recommended when calculating the uncertainty of complex-valued quantities [3].

This section describes a simple method to transform uncertainty statements from polar coordinates into a corresponding covariance matrix representing uncertainty in rectangular coordinates. The method is approximate. It is useful when the point in the complex plane, for which the uncertainty information has been given, is distant from the origin. In practice, this will often be the case. For example, the reflection coefficient of highly reflecting components, such as an RF 'short', or 'open', which have magnitudes very close to unity, will often be reported in polar coordinates.

3.1 Transformation procedure

Let $\mathbf{z} = (z_\mathrm{r}, z_\phi)$ be an estimate of a complex quantity expressed in polar coordinates, with standard uncertainties $u(z_\mathrm{r})$, for magnitude, and $u(z_\phi)$, for phase. To obtain the covariance matrix associated with $\mathbf{z}$ in the real-imaginary rectangular coordinate system, we proceed as follows:

- in a rectangular coordinate system aligned with the radial and tangential directions at $\mathbf{z}$, define a matrix[4]

$$\mathbf{V}_\mathrm{rt} = \begin{bmatrix} u(z_\mathrm{r})^2 & 0 \\ 0 & u(z_\mathrm{t})^2 \end{bmatrix} , \tag{8}$$

  where

$$u(z_\mathrm{t}) = z_\mathrm{r} \tan u(z_\phi) . \tag{9}$$

- define a matrix for the rotation, though z_ϕ, that orients this coordinate system

[4]It is often assumed that $u(z_\mathrm{r})$ and $u(z_\mathrm{t})$ are independent, so the off-diagonal elements of $\mathbf{V}_\mathrm{rt}$ are shown here as zero. In general, the off-diagonal elements are approximately equal to $r\, u(z_\mathrm{r})\, u(z_\mathrm{t})$, where r is the correlation coefficient between $u(z_\mathrm{r})$ and $u(z_\phi)$.

with the conventional real-imaginary coordinate system

$$\mathbf{R} = \begin{bmatrix} \cos z_\phi & -\sin z_\phi \\ \sin z_\phi & \cos z_\phi \end{bmatrix} \tag{10}$$

- obtain the required covariance matrix in real-imaginary coordinates by the transformation[5]

$$\mathbf{V} = \mathbf{R}\,\mathbf{V}_{\mathrm{rt}}\,\mathbf{R}'\,. \tag{11}$$

The approximate step in this procedure is equation (9). If the curvature of the arc in the complex plane representing phase uncertainty is too great the procedure will not be reliable. However, it is difficult to quantify how this approximation ultimately affects the level of confidence of subsequent uncertainty statements. This question will be examined further in the Appendix.

3.2 Example

The need to transform uncertainty statements from polar to rectangular coordinates can arise, for example, when a calibration report gives data in polar coordinates.

Suppose that reflection coefficient data for an offset short, at one frequency, are given as: $\Gamma_{\mathrm{r}} = 0.995$, $\Gamma_\phi = 85.34°$, $u(\Gamma_{\mathrm{r}}) = 0.013$ and $u(\Gamma_\phi) = 0.88°$. We can immediately write

$$\mathbf{V}_{\mathrm{rt}} = \begin{bmatrix} 1.69 \times 10^{-4} & 0 \\ 0 & 2.34 \times 10^{-4} \end{bmatrix}$$

and

$$\mathbf{R} = \begin{bmatrix} 0.081 & -0.997 \\ 0.997 & 0.081 \end{bmatrix}$$

and obtain the required covariance matrix using (11)

$$\mathbf{V} = \begin{bmatrix} 2.33 \times 10^{-4} & -5.22 \times 10^{-6} \\ -5.22 \times 10^{-6} & 1.69 \times 10^{-4} \end{bmatrix}\,.$$

The standard uncertainties of the real and imaginary components are the square roots of the diagonal elements, $u(\Gamma_{\mathrm{re}}) = 0.015$ and $u(\Gamma_{\mathrm{im}}) = 0.013$, and we see, from the non-zero off-diagonal term, that there is a small correlation coefficient $r = -0.03$.

3.3 The reverse transformation

The transforation of variance-covariance information from the real-imaginary coordinate system to polar coordinates is not recommended, as already mentioned [3]. Nevertheless, there may be situations where it will be useful to describe uncertainties in a 'radial-tangential' rectangular coordinate system. That is, when the radial and tangential directions have a clear physical interpretation at the point of interest. To do so is reasonable because the rotation of a rectangular coordinate system preserves all information about $\mathbf{V}$ in $\mathbf{V}_{\mathrm{rt}}$, and hence in $u(z_{\mathrm{r}})$, $u(z_{\mathrm{t}})$ and r.

[5]This is a standard result that applies to covariance matrices for any linear transformation of coordinates (see, e.g., [8, Ch. 2]).

The reverse transformation involves a rotation of $-z_\phi$, corresponding to the matrix $\mathbf{R}'$, so we obtain the covariance matrix in the 'radial-tangential' system of coordinates

$$\mathbf{V}_{\mathrm{rt}} = \mathbf{R}'\,\mathbf{V}\,\mathbf{R}\ . \tag{12}$$

The standard uncertainties $u(z_\mathrm{r})$ and $u(z_\mathrm{t})$, and the correlation coefficient r, between $u(z_\mathrm{r})$ and $u(z_\mathrm{t})$, are then obtained from the matrix elements

$$\begin{aligned} u(z_\mathrm{r}) &= \sqrt{\mathbf{V}_{\mathrm{rt}}(1,1)} \\ u(z_\mathrm{t}) &= \sqrt{\mathbf{V}_{\mathrm{rt}}(2,2)} \\ r &= \mathbf{V}_{\mathrm{rt}}(1,2) \,/\, [u(z_\mathrm{r})u(z_\mathrm{t})]\ . \end{aligned}$$

4 Discussion

The material presented here complements the computational tools already available to support uncertainty calculations for complex-valued quantities in RF measurements. These include: an extension of the LPU to complex (bivariate) quantities [3], convenient computational algorithms and scalar summary values for the matrix quantities involved [4] and an extension of the notion of degrees-of-freedom for type-A uncertainties [9]. This communication addresses two remaining issues: how to incorporate information about some common type-B uncertainties arising in RF measurements and how to incorporate information from uncertainty statements given in polar coordinates.

There is now, broadly speaking, a full complement of methods available to perform the kind of uncertainty calculations that arise frequently in RF metrology. Moreover, when applied to real-valued data, these bivariate methods are equivalent to those in the *Guide*. So it is entirely possible to merge parts of an uncertainty calculation that use univariate methods from the *Guide* with parts using a bivariate formulation. Indeed, situations that require this are likely to arise in practice. The example given in section 2.2 is a case in point, where a real-valued mismatch term arises in a measurement situation dominated by complex quantities.

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5 Appendix: Assessing the accuracy of the polar to rectangular transformation

It is difficult to quantify the effect that the approximation of equation (9) has on the level of confidence of subsequent uncertainty statements. That is, whether an uncertainty region, obtained using covariance data from transformed statements of uncertainty in polar coordinates, provides the expected level of confidence.

This is a question about the performance of the *procedure* used to calculate uncertainty regions, which includes the polar to rectangular transformation. If, over many

independent repetitions of an experiment, the calculated uncertainty regions contain the measurand at a rate close to the desired level of confidence, then the procedure is acceptable.

It is possible to test a procedure to determine its validity in a particular situation, using a simulation method. We proceed as follows

1. Choose a complex value in polar coordinates $\mu = (\mu_{\rm r}, \mu_\phi)$ that is representative of the measurand being considered.

2. Choose standard errors $e_{\rm r}$ and e_ϕ, that are representative of the uncertainty of the measurement procedure being considered.

3. Using random number generators, obtain $z_{\rm r}$ from a Gaussian distribution with mean $\mu_{\rm r}$ and standard deviation $e_{\rm r}$. Similarly, obtain z_ϕ from a Gaussian distribution with mean μ_ϕ and standard deviation e_ϕ.

4. Apply the transformation procedure of section 3.1, using $\mathbf{z} = (z_{\rm r}, z_\phi)$ with $u(z_{\rm r}) = e_{\rm r}$ and $u(z_\phi) = e_\phi$ as the standard uncertainties[6], to calculate the covariance matrix $\mathbf{V}$.

5. Obtain a region of uncertainty for μ, with a $100p\%$ level of confidence, and test whether μ is contained within it. That is, evaluate the 'statistical' (Mahalanobis) distance [7]
$$d^2 = (\mathbf{z} - \mu)' \, \mathbf{V}^{-1} \, (\mathbf{z} - \mu) \tag{13}$$
and consider that the region contains μ when
$$d^2 \leq \chi^2_{2,p} \, ,$$
where $\chi^2_{2,p}$ is the $100p\%$ point of the chi-square distribution with two degrees of freedom.[8] (Note, the relation $\chi^2_{2,p} = -2\ln(1-p)$ may be useful in calculations.)

6. Repeat steps 4–5 many times, recording the rate of success at step 5.

In situations where the procedure is acceptable, we expect the success-rate to be close to the desired level of confidence after a large number of simulated experiments.

5.1 Example

A short program, reproduced below, has been used to check the validity of the polar-to-rectangular conversion for the example in section 3.2. The program performs 10000 repetitions of a simulated experiment and reports the number of successes. A typical result obtained was 9474, or a 94.74% success rate.[9]

[6] This corresponds to the case of infinite degrees-of-freedom. A finite degrees-of-freedom calculation could, for example, generate a sample of values of $z_{\rm r}$ and z_ϕ and use sample statistics to determine the best estimate and standard uncertainties required at this step.

[7] For this calculation, complex quantities are represented as 2-element column vectors, containing the real and imaginary components.

[8] If the calculation involves finite degrees-of-freedom the inequality becomes $d^2 \leq 2\nu/(\nu-1)\, F_{2,\nu-1,p}$, where $F_{2,\nu-1,p}$ is the $100p\%$ percentage point of the Fisher-Snedecor distribution and ν is the number of degrees of freedom.

[9] The 10000 size of the simulation in this case means that the standard deviation for the number of successes observed will be about 21.

It is interesting that the transformation in this case is fairly sensitive to the values of uncertainty. While keeping e_{r} constant, we increased e_ϕ first to $2e_\phi$, and then to $5e_\phi$, and observed corresponding simulation success-rates fall to 94%, and then to 84%, respectively. However, the success rates were restored to essentially 95% in both cases if the value of e_{r} was also scaled by the same amount.

5.1.1 Program listing

The following program uses R software [10]. Text on a line that is preceeded by the symbol # is a comment.

```
# Write the value and uncertainty here
# NB polar angles must be in radians
mu_polar <- c(0.995,1.489)
mu_std_e <- c(0.013,0.015)

# Calculate mu in rectangular coordinates
mu <- c(mu_polar[1] * cos( mu_polar[2] ),
    mu_polar[1] * sin( mu_polar[2] ))

# Simulated data for N experiments
N <- 10000
z_polar <- cbind(
    rnorm(N,mu_polar[1],mu_std_e[1]),
        rnorm(N,mu_polar[2],mu_std_e[2]))

# Find the number of 'successful' uncertainty regions
chi_sqr_p <- qchisq(0.95,2) # A constant
ok <- rep(1,N)          # individual results
for( i in 1:length(ok) ) {
    r <- z_polar[i,1]
    phi <- z_polar[i,2]
    z <- c(r * cos( phi ),r * sin( phi ))

    V_rt <- matrix(
        c(mu_std_e[1]^2, 0,
            0, (r * tan(mu_std_e[2]))^2),
                2,2)

    R <- matrix(
        c(cos(phi),-sin(phi),sin(phi),cos(phi)),
            2,2)

    V <- R %*% V_rt %*% t(R)        # Eqn (11)
    d_sqr <- mahalanobis(z,mu,V)    # Eqn (12)

    ok[i] <- (d_sqr <= chi_sqr_p)   # 0 or 1
}
print( sum(ok) )
```

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