Industrial Research Limited Report 2557

Notes on complex measurement uncertainty – part 2

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28 September 2012 (replaces 19 Jan 2012)

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Reference

B. D. Hall; 28 September 2012 (replaces 19 Jan 2012). Notes on complex measurement uncertainty – part 2. Industrial Research Limited Report 2557; Lower Hutt, New Zealand.

Summary

This report describes methods of evaluating measurement uncertainty in complex quantities with a particular emphasis on RF and microwave measurements. The intent is to provide a concise reference.

The report has been prepared in two parts. Part 2 (this document) discusses a general method of treating complex measurement uncertainty that can be used when the measurand is complex-valued and influence quantities may be correlated.

Part 1 described a simpler approach that could be used only when rather strict conditions apply, namely: all influence quantities are mutually independent and the uncertainties in the real and imaginary components of each influence quantity are equal. In that case, the uncertainty region can be considered a circle and the mathematical methods used to propagate uncertainty are not as complicated than those presented here.[1]

This document is divided into a number of sections covering

- the statement of uncertainty for a complex quantity
- statistical evaluation of uncertainty (type-A)
- evaluation of uncertainty from other information (type-B)
- propagation of uncertainty
- degrees of freedom

Each section begins with a summary of the particular topic, which is followed by subsections with supplementary information and comments. A few software examples are given at the end of each section.

[1] B D Hall, Notes on complex measurement uncertainty – part 1, Industrial Research Report 2483, November 2010. (`http://rf.irl.cri.nz/Documents_central#IRL_reports_list`)

Contents

1 Introduction

The evaluation and expression of uncertainty in measurements of complex quantities is not fully covered in the *Guide to the Expression of Uncertainty in Measurement* (GUM) [1]. So, part of the international metrology community interested in radio frequency (RF) and microwave measurements have developed methods of working with uncertainty in complex quantities, while trying to respect the approach taken in the GUM. This report assembles information from various sources about this topic.

The full report is in two parts. Part 2 (this document) deals with a general treatment of the complex measurement uncertainty problem. It is assumed that the measurand and all influence quantities are complex-valued and that there may be correlation between estimates of quantities that influence the measurement.

Part 1 described a simpler approach that could be used only when rather strict conditions apply, namely: all influence quantities are mutually independent and when the uncertainties in the real and imaginary components of each influence quantity are equal [2]. In that case, the uncertainty region can be considered a circle and the mathematical methods used to propagate uncertainty are not as complicated than those presented here.

1.1 Using this document

This report is primarily for reference. Sections cover distinct topics and, although there is some cross-referencing, they could be read in any order.

Here is a short summary of the section contents.

Uncertainty statements for complex quantities introduces the notion of an uncertainty region in the complex plane. Additional comments include: different expressions of uncertainty as a region; the meaning of coverage probability and the difference between real and complex coverage factors.

Evaluating type-A uncertainty describes the conventional statistical method for evaluating uncertainty using sample statistics.

Evaluating type-B uncertainty describes how uncertainty can be evaluated when the phase of a complex quantity is unknown. Additional comments relate to the notion of degrees-of-freedom for a type-B uncertainty and the validity of associating non-Gaussian distributions with measurement errors.

Propagation of uncertainty describes a method of propagating uncertainty for complex quantities. Additional comments relate to: alternative formulations of the method, including application of the GUM method to the complex problem; simplifying partial derivative calculations and summarising matrix quantities that arise in these calculations.

Degrees of freedom describes how to evaluate the effective degrees-of-freedom associated with a complex uncertainty region. Additional comments relate to the importance of degrees of freedom in obtaining accurate uncertainty statements and an approximate expression for the effective degrees of freedom for type-B uncertainty

1.2 Software

The method of evaluating uncertainty described in this document leads to more complicated computations than the method of Part 1. This is unavoidable. Three terms are required to describe the uncertainty of a complex quantity: the standard uncertainty of the real and imaginary components and the correlation between them. Manipulation of these terms requires extra work.

Fortunately, a large number of repetitive computational steps can be automated. So it seems appropriate to show how software tools can facilitate uncertainty calculations. This report includes short examples using a tool called the *GUM Tree Calculator* (GTC), which runs under Microsoft Windows. GTC can be downloaded from `http://mst.irl.cri.nz`.[2] The software examples in this document use GTC version 0.9.7.

1.3 Notation

Real-valued quantities and quantity estimates are written in plain italic font, like X or x. Complex-valued quantities and quantity estimates are written in bold italic font, like $\boldsymbol{X}$ or $\boldsymbol{x}$. Greek characters are not italicised. They are shown in bold when representing complex values, e.g., Γ and $\boldsymbol{\Gamma}$.

Matrices are typeset in bold Roman (upright) font, like $\mathbf{v}$.

The imaginary unit j is used, where $\mathrm{j}^2 = -1$ (e.g., $\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}\, x_{\mathrm{im}}$).

When describing mathematical methods in general, uppercase characters represent physical quantities (that have a fixed although unknown value) and lowercase characters denote quantity estimates. However, because this notation generally clashes with conventions in many measurement domains, the distinction between quantities and estimates is determined by the context when describing measurement examples.

1.4 Change history

19 Jan 2012: initial release uses `GTC` software version 0.9.6

28 Sept 2012: updated listings to use `GTC` version 0.9.7

1.4.1 Changes made in the Sept 2012 version

- Two of the function definitions for the effective reflection coefficient in §5.3.1 were incorrect, leading to incorrect calculated values.
- A bug in the `summary()` function of `GTC` version 0.9.6 has been fixed. The value of the correlation coefficient (`r`) between the real and imaginary components of an uncertain complex number was sometimes incorrectly displayed as zero.

  Note that the function `get_correlation()` returned the correct value. However, the `summary()` string sometimes incorrectly displayed `'r = 0'`.

[2] Registration is required, but the software is free. It can be found in the `Downloads` menu under the item `GUM Tree Calculator`. The software is easily installed. Detailed documentation is provided.

2 Uncertainty statements for complex quantities

2.1 Main points

The measurement uncertainty of a complex quantity is represented by a region in the complex plane.

Figure 1 shows an uncertainty region for a complex quantity $\boldsymbol{X}$ estimated by

$$\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}x_{\mathrm{im}} \ .$$

The uncertainty ellipse is centered on $\boldsymbol{x}$ and scaled by the coverage factor, $k_{2,p}$, to provide approximately $100p$ % coverage probability, or level of confidence.

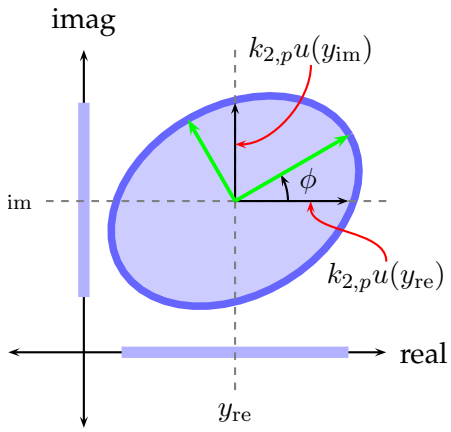

Figure 1: An uncertainty region for a complex quantity.

The shape of the region depends on the standard uncertainty of the real and imaginary component estimates, $u(x_{\mathrm{re}})$ and $u(x_{\mathrm{im}})$, and on the correlation between them r.

When $r = 0$, the ellipse aligns with the real and imaginary coordinate axes, as indicated in Figure 2, and the lengths of the ellipse axes are proportional to the standard uncertainties of the real and imaginary components.

When $r \neq 0$, the ellipse is oriented at an angle to the coordinate axes (as shown in Figure 1).

A statement of uncertainty should provide information about the standard uncertainties of the real and imaginary components $u(x_{\mathrm{re}})$ and $u(x_{\mathrm{im}})$, and the correlation r between them. A covariance matrix contains all this information

$$\mathbf{v}(\boldsymbol{x}) = \begin{bmatrix} u(x_{\mathrm{re}})^2 & u(x_{\mathrm{re}})\, r\, u(x_{\mathrm{im}}) \\ u(x_{\mathrm{re}})\, r\, u(x_{\mathrm{im}}) & u(x_{\mathrm{im}})^2 \end{bmatrix} \ . \tag{1}$$

A coverage factor $k_{2,p}$ is used to scale the uncertainty ellipse. It is a function of the required coverage probability and the number of degrees of freedom ν (discussed in §3.1 and §6.1)[3]

$$k_{2,p}^2 = \frac{2\nu}{\nu - 1} F_{2,\nu-1}(p) \ , \tag{2}$$

[3]This coverage factor is also mentioned briefly in §6.5.4 of [3].

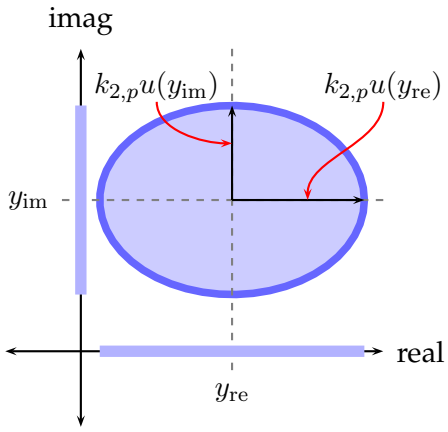

Figure 2: The axes of an uncertainty ellipse are parallel to the real and imaginary coordinate axexs when the estimates x_{re} and x_{im} are independent $(r = 0)$.

where $F_{2,\nu-1}(p)$ is the upper $100p^{\mathrm{th}}$ percentile of the F-distribution [4, Ch 5].

When ν is infinite

$$k_{2,p}^2 = \chi_{2,p}^2 \ , \tag{3}$$

where $\chi_{2,p}^2$ is the $100p\%$ point of the chi-square distribution with 2 degrees of freedom.[4][5]

2.2 Additional comments

2.2.1 Uncertainty statements for real quantities

The measurement uncertainty of a real-valued quantity is an interval.

A measurand is considered to lie somewhere between the upper and lower bounds of the uncertainty interval, with a certain coverage probability.

The half-width of the interval is called the *expanded uncertainty*

$$U = k_p u \ ,$$

where k_p is the one-dimensional coverage factor, p is the required coverage probability and u is the standard uncertainty associated with an estimate of the quantity of interest. If x is the estimate of the quantity of interest, the uncertainty interval is

$$[x - U, x + U] \ .$$

2.2.2 Describing an elliptical uncertainty region

Unfortunately, there seems to be no simple way to describe elliptical uncertainty regions, nor to determine when a point in the complex plane lies inside a region.

A calculation called the *Mahalanobis Distance* can be used to describe the extent of an uncertainty region.

[4] The relation $\chi_{2,p}^2 = -2\log_{\mathrm{e}}(1-p)$ may be useful for calculations.
[5] The case of $\nu = \infty$ is also mentioned in §6.5.3 of [3]

The squared Mahalanobis distance between two points $\boldsymbol{\xi}$ and $\boldsymbol{x}$ in the complex plane is

$$d(\boldsymbol{x},\boldsymbol{\xi})^2 = (\boldsymbol{\xi}-\boldsymbol{x})^{\mathrm{T}}\mathbf{v}(\boldsymbol{x})^{-1}(\boldsymbol{\xi}-\boldsymbol{x}) \ ,$$

where $\mathbf{v}(\boldsymbol{x})^{-1}$ is the inverse covariance matrix and $\boldsymbol{\xi} = (\xi_{\mathrm{re}}, \xi_{\mathrm{im}})^{\mathrm{T}}$ and $\boldsymbol{x} = (x_{\mathrm{re}}, x_{\mathrm{im}})^{\mathrm{T}}$ are treated as vectors for the calculation.

The point $\boldsymbol{\xi}$ then lies inside an uncertainty region associated with $\boldsymbol{x}$ when

$$d(\boldsymbol{x},\boldsymbol{\xi})^2 \leq k_{2,p}^2 \ .$$

The locus of all points $\boldsymbol{\xi}$ satisfying this inequality defines the uncertainty region for $\boldsymbol{x}$.

2.2.3 Simultaneous uncertainty intervals

A pair of uncertainty intervals, one for the real component and one for the imaginary component, provide an alternative and more intuitive rectangular uncertainty region.

There are two ways of constructing simultaneous uncertainty-intervals, which we will call: T^2*-intervals* and *Bonferroni intervals*.

> **Note** the T^2-intervals and Bonferroni intervals describe rectangular regions in the complex plane that have a probability greater than or equal to $100\,p\%$ of containing $\boldsymbol{X}$. The individual intervals are not independent uncertainty statements for the real and imaginary components (which would be constructed using the one-dimensional coverage factor k_p, as described in §2.2.1).

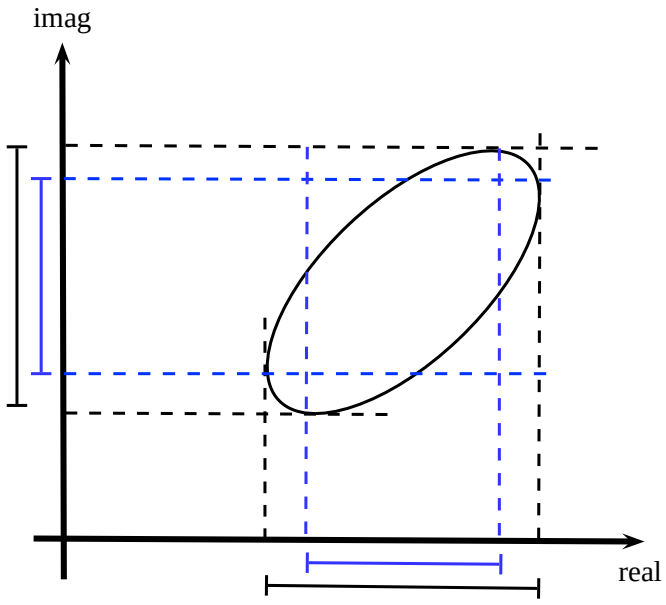

Figure 3: Relationship of rectangular uncertainty regions to the uncertainty ellipse with the same coverage probability. The (wider) black bars represent T^2 intervals, the (narrower) blue bars represent Bonferroni intervals.

T^2-intervals are the projection of the uncertainty ellipse onto the real, or imaginary, axis (see Fig 3 and the blue bands along the axes in Figs 1 and 2).

Expanded uncertainties for T^2 intervals are calculated using the two-dimensional coverage factor. For a $100\,p\%$ coverage probability,

$$\begin{aligned} U_{\mathrm{re}} &= k_{2,p}\, u(x_{\mathrm{re}}) \\ U_{\mathrm{im}} &= k_{2,p}\, u(x_{\mathrm{im}})\,, \end{aligned}$$

which leads to a pair of simultaneous uncertainty intervals for the real and imaginary components of $\boldsymbol{X}$

$$\begin{aligned} &[x_{\mathrm{re}} - U_{\mathrm{re}}, x_{\mathrm{re}} + U_{\mathrm{re}}] \\ &[x_{\mathrm{im}} - U_{\mathrm{im}}, x_{\mathrm{im}} + U_{\mathrm{im}}]\,. \end{aligned}$$

Bonferroni intervals are always narrower than the corresponding T^2-intervals (see Fig 3) and hence provide a more precise estimate [4, §5.4].[6]

The calculation for Bonferroni intervals uses the one-dimensional coverage factor with a modified probability parameter

$$\begin{aligned} p_{\mathrm{B}} &= 1 - (1-p)/2 \\ &= (1+p)/2\,. \end{aligned}$$

So,

$$\begin{aligned} U_{\mathrm{re}} &= k_{p_{\mathrm{B}}}\, u(x_{\mathrm{re}}) \\ U_{\mathrm{im}} &= k_{p_{\mathrm{B}}}\, u(x_{\mathrm{im}})\,. \end{aligned}$$

Although narrower than T^2 intervals, the coverage probability of a Bonferroni uncertainty region is still greater than or equal to $100\,p\%$.

2.2.4 What does coverage probability mean?

Coverage probability, or level of confidence, is a performance measure of the method used to calculate uncertainty.

Measurement errors are unpredictable, so there can be no guarantee that an uncertainty interval, or region, covers the measurand in a particular case. Nevertheless, the success-rate of a method of uncertainty calculation can be specified with respect to a large number of hypothetical independent measurements.

For example, an uncertainty calculation with a nominal 95% coverage probability should generate uncertainty statements that cover the measurand on about 95 out of 100 occasions.

It is appropriate to think of the stated coverage probability as the probability that the *method* of calculating the uncertainty region, or interval, generates a statement containing the measurand.

[6] Although not mentioned by name, Bonferroni intervals are described briefly in §6.5.2.3 b) of [3].

2.2.5 What is the difference between real and complex coverage factors?

The 1-D coverage factor k_p is used for real-valued problems, where an uncertainty interval is required, and the 2-D factor $k_{2,p}$ is used for complex-valued problems, where an uncertainty region is required. In both cases, the coverage factor scales the standard uncertainty. So the area of an uncertainty region is proportional to $k_{2,p}^2$.

The value of coverage factors $k_{2,p}$ and k_p for the same degrees of freedom and the same probability p are different. Table 1 compares the two coverage factors at 95% coverage probability and different degrees of freedom. Comments in §6.2.2 are also relevant.

Table 1: A comparison of 1-D and 2-D coverage factors for $p = 0.95$.

ν	$k_{2,0.95}$	$k_{0.95}$	ν	$k_{2,0.95}$	$k_{0.95}$
2	28.3	4.3	7	3.5	2.4
3	7.6	3.2	8	3.3	2.3
4	5.1	2.8	9	3.2	2.3
5	4.2	2.6	10	3.1	2.2
6	3.7	2.5	50	2.6	2.0
			∞	2.45	1.96

2.2.6 Can the covariance matrix be summarised as a magnitude?

The mean of the diagonal terms (the standard variance in the real and imaginary components) is a useful summary value for a covariance matrix

$$\overline{v} = \frac{v_{11} + v_{22}}{2} \, .$$

Several properties of $\overline{v}$ are worth noting:

- $\overline{v}$ depends on the size and shape of the ellipse, but not the orientation.
- $\sqrt{\overline{v}}$ has the units of the measurand and is characteristic of the uncertainty in the components in the absence of correlation.
- If the covariance obtained is by a type-A evaluation of uncertainty (described in §3.1), $\sqrt{\overline{v}}$ is the radius of a circular uncertainty region, which is equivalent to the region described in Part 1 of this report [2, §4.1].
- $\overline{v}$ is proportional to the mean area of the inscribed and circumscribed circles of the ellipse (see Figure 4).

2.2.7 Polar coordinates

Polar coordinates are often used when reporting measurement results. Unfortunately, their use can lead to problems with the calculation of uncertainty. For that reason, polar coordinates should be avoided when reporting uncertainty if possible (see also §3.2.2).

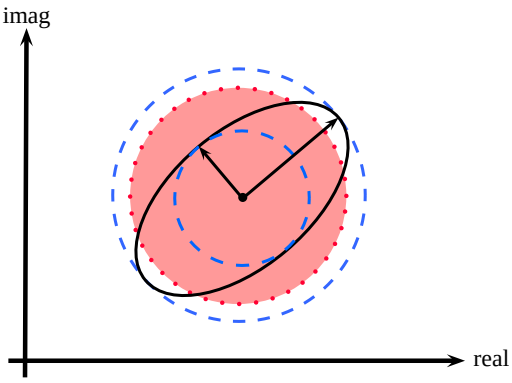

Figure 4: $\overline{v}$ is proportional to the mean area of the inscribed and circumscribed circles.

However, the following method can be used to convert uncertainty statements from polar to rectangular coordinates, provided the uncertainty in the angular coordinate is not too large (see also [5, 6]).

Given $\boldsymbol{x}$ in polar coordinates (x_r, x_ϕ) with standard uncertainties $u(x_\mathrm{r})$ and $u(x_\phi)$

- define a 'radial-tangential' Cartesian coordinate system, aligned with the polar directions at $\boldsymbol{x}$ (the r-t coordinates in Fig 5)
- express the uncertainty in the tangential coordinate as

$$u(x_\mathrm{t}) = x_\mathrm{r} u(x_\phi)$$

- transform the covariance matrix

$$\mathbf{v}_\mathrm{rt} = \begin{bmatrix} u(x_\mathrm{r})^2 & 0 \\ 0 & u(x_\mathrm{t})^2 \end{bmatrix}$$

by rotating it

$$\mathbf{v} = \Phi\, \mathbf{v}_\mathrm{rt}\, \Phi'$$

with the matrix

$$\Phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} .$$

The resulting covariance matrix $\mathbf{v}$ is in rectangular coordinates (Fig 6).

2.3 Software

2.3.1 Estimates and uncertain numbers

GTC software uses internal constructs for real and complex quantity estimates. These are called *uncertain numbers*.

To define an uncertain real number representing an estimate $x = 11$, with an uncertainty $u(x) = 1.5$ and degrees of freedom = 12, we write

```
>>> x = ureal(11,1.5,12)
```

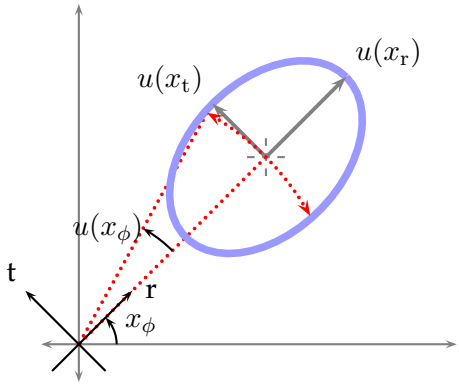

Figure 5: Radial-tangential Cartesian coordinates align with the polar coordinate directions of $\boldsymbol{x}$.

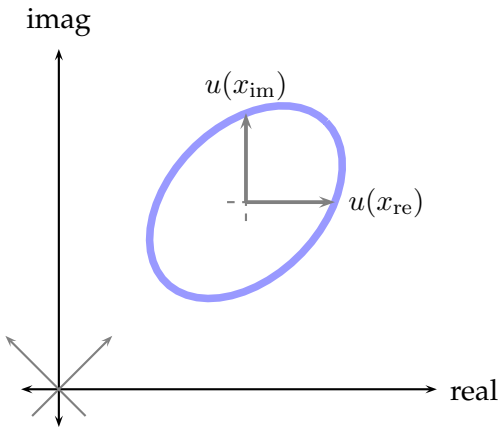

Figure 6: After rotation, the covariance matrix describes the uncertainty in rectangular coordinates.

To define an uncertain complex number representing an estimate $\boldsymbol{z} = 1.1 + \mathrm{j}0.3$, with uncertainties $u_\mathrm{re}(\boldsymbol{z}) = 0.05$, $u_\mathrm{im}(\boldsymbol{z}) = 0.25$ and degrees of freedom = 25, we could write[7]

```
>>> z = ucomplex(1.1+0.3j,(0.05,0.25),25)
```

Once defined, uncertain numbers can be used much like ordinary numbers in mathematical expressions.

```
>>> y = sqrt(x)
>>> y
ureal(3.3166247903554, 0.22613350843332272, 12.0)
```

The value, uncertainty and degrees of freedom of an uncertain number can be obtained using attributes, or by applying functions. For instance,

```
>>> x = ureal(11,1.5,12)
>>> x.x, x.u, x.df
(11.0, 1.5, 12)
```

[7] There are alternative ways to specify the uncertainty of an uncertain complex number, see GTC documentation for details.

```
>>> value(x), uncertainty(x), dof(x)
(11.0, 1.5, 12)

>>> y.x, y.u, y.df
(3.3166247903554, 0.22613350843332272, 12.0)
```

2.3.2 Coverage factors

Functions defined in the GTC `reporting` module calculate coverage factors.

For real-valued problems, `reporting.k_factor` is used. For example,

```
>>> x = ureal(11,1.5,12)
>>> reporting.k_factor(x.df) * x.u
3.268219244501992
```

For complex problems, `reporting.k2_factor_sq` returns the coverage factor squared (see §2.3.3). For example, the coverage factors in the $k_{2,0.95}$ column of Table 1 can be obtained by

```
>>> dof = [2,3,4,5,6,7,8,9,10,50,inf]
>>> for df in dof:
...  print math.sqrt( reporting.k2_factor_sq(df) )
...
28.2488937836
7.54983443523
5.04700426081
4.16661490601
3.72648951482
3.46423488178
3.29064965458
3.16744104001
3.07552876361
2.55014299414
2.44774683068
```

2.3.3 Mahalanobis distance

Suppose a measurement yields $\boldsymbol{x} = 1.3 - \mathrm{j}0.87$, with $u(x_{\mathrm{re}}) = 0.01$, $u(x_{\mathrm{im}}) = 0.02$, $r = -0.1$ and $\nu = 5$. We wish to know whether the points $\boldsymbol{\xi}_1 = 1.3 - \mathrm{j}0.95$ and $\boldsymbol{\xi}_2 = 1.35 - \mathrm{j}0.87$ lie inside the uncertainty region associated with $\boldsymbol{x}$. We proceed as follows.

First, we can define an uncertain number representing $\boldsymbol{x}$. The required covariance matrix is conveniently calculated by `u_to_cv`, from the `reporting` module.

```
>>> cv = reporting.u_to_cv( (0.01,0.02), -0.1 )
>>> x = ucomplex(1.3-0.87j,cv,5)
>>> x
ucomplex( (1.3-0.87j), [0.0001,-2e-05,-2e-05,0.0004], 5.0 )
```

Then we can check the points $\boldsymbol{\xi}_1 = 1.3 - \mathrm{j}0.95$ and $\boldsymbol{\xi}_2 = 1.35 - \mathrm{j}0.87$ as follows

```
>>> xi_1 = 1.3 - 0.95j
>>> xi_2 = 1.35 - 0.87j
>>> k2_sq = reporting.k2_factor_sq(x.df)
>>> reporting.mahalanobis_sq(x.x,xi_1,x.v) <= k2_sq
True
>>> reporting.mahalanobis_sq(x.x,xi_2,x.v) <= k2_sq
False
```

By testing whether the squared Mahalanobis distances are less than the squared coverage factor, we see that $\boldsymbol{\xi}_1$ is inside the uncertainty region but $\boldsymbol{\xi}_2$ is not. Note that, if the covariance matrix is ignored, $\boldsymbol{\xi}_2$ actually appears to be closer to $\boldsymbol{x}$ than $\boldsymbol{\xi}_1$.

```
>>> abs(x-xi_1)
0.07999999999999996
>>> abs(x-xi_2)
0.050000000000000044
```

2.3.4 Bonferroni intervals

Suppose a measurement yields $\boldsymbol{x} = 1.3 - \mathrm{j}0.87$, with $u(x_{\mathrm{re}}) = 0.01$, $u(x_{\mathrm{im}}) = 0.02$, $r = -0.1$ and $\nu = 5$. What are the 95% Bonferroni intervals?

The coverage factor can be calculated and then the expanded uncertainties as follows

```
>>> x_re, x_im = 1.3,-0.87
>>> u_re, u_im = 0.01, 0.02
>>> p_B = 100*(1+0.95)/2 # in percent
>>> k = reporting.k_factor(5,p_B)
>>> U_re = k*u_re
>>> U_im = k*u_im
```

Finally, the uncertainty intervals are defined here as Python lists

```
>>> interval_re = [x_re-U_re,x_re+U_re]
>>> interval_im = [x_im-U_im,x_im+U_im]
>>> interval_re
[1.2683661854989066, 1.3316338145010935]
>>> interval_im
[-0.9332676290021871, -0.8067323709978129]
```

If we again ask: "do the points $\boldsymbol{\xi}_1 = 1.3 - \mathrm{j}0.95$ and $\boldsymbol{\xi}_2 = 1.35 - \mathrm{j}0.87$ fall inside the uncertainty region associated with $\boldsymbol{x}$?", we can easily test this. To be inside the region, the coordinate must be inside both uncertainty intervals. So we define a function that does an interval check[8]

```
test = lambda x,interval: interval[0] <= x and x <= interval[1]
```

Applying this to the data

[8] This is an instance of a Python 'lambda' function, which is a concise way of defining short functions. The function arguments are `x`, the point of interest, and `interval`, a 2-element sequence with the lower and upper interval limits. The function `test` returns `True` if `x` is in the interval and `False` otherwise.

```
>>> xi_1 = 1.3 - 0.95j
>>> xi_2 = 1.35 - 0.87j
>>> test(xi_1.real,interval_re) and test(xi_1.imag,interval_im)
False
>>> test(xi_2.real,interval_re) and test(xi_2.imag,interval_im)
False
```

we see that neither point is inside the Bonferroni rectangle. This is surprising, because the Mahalanobis calculation in §2.3.3 placed ξ_1 inside the uncertainty ellipse.

The different result for the Bonferroni intervals is not an error. The Bonferroni and elliptical uncertainty regions cover slightly different parts of the complex plane. The Bonferroni intervals are narrower than projections of the uncertainty ellipse onto the real and imaginary axes, so some points near the perimeter of the ellipse lie outside the uncertainty rectangle. There are also points inside the Bonferroni rectangle that fall outside the ellipse. Overall, the coverage of the Bonferroni uncertainty region will be at least 95%.

Had T^2 intervals been calculated, ξ_1 would lie inside the uncertainty region and ξ_2 outside. To do this in GTC, calculate the coverage factor `k` in the following way. (The rest of the calculation is the same).

```
>>> k2 = reporting.k2_factor_sq(5)
>>> k = math.sqrt(k2)
```

2.3.5 Mean variance

In this example, we show that the mean variance $\overline{v}$ does not change when an uncertainty region is rotated.

First, an uncertain complex number is created with $\boldsymbol{z} = 10.1 + \mathrm{j}3.3$ and covariance

$$\mathbf{v}(\boldsymbol{z}) = \begin{bmatrix} 1 & -0.1 - 0.1 & 0.5 \end{bmatrix} .$$

```
>>> cv = (1,-0.1,-0.1,0.5)
>>> z = ucomplex(10.1 + 3.3j,cv)
>>> z.v
variance_covariance(rr=1.0, ri=-0.1, ir=-0.1, ii=0.5000000000000001)
>>> reporting.v_bar( z.v )
0.75
```

The mean variance $\overline{v}(\boldsymbol{x}) = 0.75$.

The orientation of the associated uncertainty region is rotated when `z` is multiplied by $e^{\mathrm{j}\phi}$. Here we will apply a rotation of $\phi = 30$ degrees.

```
>>> rotate_30 = cmath.rect( 1, math.radians(30) )
>>> z *= rotate_30
>>> z.v
variance_covariance(rr=0.961602540378444, ri=0.1665063509461096, ir
    =0.1665063509461096, ii=0.5383974596215562)
>>> reporting.v_bar( z.v )
```

```
0.75
```

After two more rotations, `z` will have been turned a total of 90 degrees. We see below that the real and imaginary component variances are interchanged, but the mean variance remains the same.

```
>>> z *= rotate_30
>>> z *= rotate_30
>>> z.v
variance_covariance(rr=0.49999999999999994, ri=0.10000000000000016, ir
    =0.10000000000000016, ii=1.0)
>>> reporting.v_bar( z.v )
0.75
```

2.3.6 Polar-to-rectangular transformation

A calibration report describes the reflection coefficient of a short as

$$\Gamma_\mathrm{r} = 0.995,\ u(\Gamma_\mathrm{r}) = 0.013$$
$$\Gamma_\phi = 85.34^\circ,\ u(\Gamma_\phi) = 0.88^\circ$$

To obtain uncertainties in rectangular coordinates, as well as the correlation coefficient, we proceed as follows

```
>>> z = cmath.rect(0.995,math.radians(85.34))
>>> u_ri, r = reporting.u_polar_to_rect( z, (0.013,math.radians(0.88)) )
>>> u_ri
standard_uncertainty(real=0.015268158501270085, imag=0.013016374532001607)
>>> r
0.026297629289617094
```

The tuple `u_ri` contains a pair of standard uncertainties for the real and imaginary component. `u_polar_to_rect` also returns the correlation coefficient r.

2.3.7 Uncertainty statements in log-polar coordinates

On some occasions parameter tolerances are described using log-polar units, with the radial component in logarithmic units and the polar coordinate in degrees, or radians.

Sometimes this may be re-interpreted as relative uncertainty for a complex quantity. To do so, the information about the radial component, u_dB, should be converted into linear units, as a relative uncertainty

$$\frac{u(x_\mathrm{r})}{x_\mathrm{r}} = 10^{u_\mathrm{dB}/20} - 1\ .$$

and the information about the polar component should be converted to radians. Then, we can associate these uncertainties with a unity-valued estimate.

For example, the corrected system performance for particular VNA reflection measurements is reported in dB (for magnitude) and in degrees (for phase): for measurements in the vicinity of $|\mathbf{\Gamma}| = 0.1$, we have $u(|\mathbf{\Gamma}|) = 0.2\,\mathrm{dB}$ and $u(\phi) = 1^\circ$.

The relative uncertainty can be defined as a GTC term

```
>>> u_x_r = (10**(0.2/20) - 1)
>>> u_x_t = math.radians(1)
>>> x = ucomplex(1,(u_x_r,u_x_t))
```

Now, suppose we measure $\boldsymbol{\gamma} = 0.09 + \mathrm{j}0.01$. An uncertain number for the measurement result is just the product $\boldsymbol{x\Gamma}$

```
>>> gamma = 0.09+ 0.01j
>>> uncertainty( x * gamma )
standard_uncertainty(real=0.0021036221157917025, imag=0.0015879727482584411)
```

3 Evaluating type A uncertainty

3.1 Main points

The best estimate of a complex quantity $\boldsymbol{X}$, based on a sample of N observations $\boldsymbol{x}_1 \cdots \boldsymbol{x}_N$, is usually the arithmetic mean (written here without the 'bar' notation)

$$\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}\, x_{\mathrm{im}} \ ,$$

where

$$x_{\mathrm{re}} = \frac{1}{N}\sum_{i=1}^{N} x_{\mathrm{re}\cdot i} \ , \quad x_{\mathrm{im}} = \frac{1}{N}\sum_{i=1}^{N} x_{\mathrm{im}\cdot i}$$

and the individual observations are

$$\boldsymbol{x}_i = x_{\mathrm{re}\cdot i} + \mathrm{j}\, x_{\mathrm{im}\cdot i} \ .$$

The standard uncertainty associated with the real componentis

$$u(x_{\mathrm{re}}) = \sqrt{\frac{1}{N(N-1)}\sum_{i=1}^{N}(x_{\mathrm{re}} - x_{\mathrm{re}\cdot i})^2} \ , \tag{4}$$

and the standard uncertainty associated with the imaginary component is

$$u(x_{\mathrm{im}}) = \sqrt{\frac{1}{N(N-1)}\sum_{i=1}^{N}(x_{\mathrm{im}} - x_{\mathrm{im}\cdot i})^2} \ . \tag{5}$$

The covariance between the real and imaginary components is

$$u(x_{\mathrm{re}}, x_{\mathrm{im}}) = \frac{1}{N(N-1)}\sum_{i=1}^{N}(x_{\mathrm{re}} - x_{\mathrm{re}\cdot i})(x_{\mathrm{im}} - x_{\mathrm{im}\cdot i}) \tag{6}$$

from which the sample correlation coefficient can be obtained

$$r = \frac{u(x_{\mathrm{re}}, x_{\mathrm{im}})}{u(x_{\mathrm{re}})\, u(x_{\mathrm{im}})} \ .$$

A variance-covariance matrix conveniently represents this information

$$\mathbf{v}(\boldsymbol{x}) = \begin{bmatrix} u(x_{\mathrm{re}})^2 & u(x_{\mathrm{re}}, x_{\mathrm{im}}) \\ u(x_{\mathrm{re}}, x_{\mathrm{im}}) & u(x_{\mathrm{im}})^2 \end{bmatrix}$$

and can always be factorised into a perhaps more intuitive form

$$\mathbf{v}(\boldsymbol{x}) = \mathbf{u}(\boldsymbol{x})\, \mathbf{r}\, \mathbf{u}(\boldsymbol{x}) \ ,$$

where

$$\mathbf{u}(\boldsymbol{x}) = \begin{bmatrix} u(x_{\mathrm{re}}) & 0 \\ 0 & u(x_{\mathrm{im}}) \end{bmatrix}$$

is a component of uncertainty matrix with the standard uncertainties along the diagonal (see §5.1) and

$$\mathbf{r} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$$

contains the correlation coefficient.

The number of degrees-of-freedom associated with $\boldsymbol{x}$ is

$$\nu = N - 1 \ . \tag{7}$$

The values of $\boldsymbol{x}$, $\mathbf{v}(\boldsymbol{x})$ and ν, characterise a measurement result and allow an uncertainty region to be evaluated. Alternatively, $\mathbf{u}(\boldsymbol{x})$ and r may be used instead of $\mathbf{v}(\boldsymbol{x})$.

3.2 Additional comments

3.2.1 Type A uncertainty for real quantities

As described in the GUM [1, §4.2.1], the best estimate of a quantity of interest X, based on a sample of N observations $x_1 \cdots x_N$, is usually the arithmetic mean (written here without the 'bar' notation)

$$x = \frac{1}{N} \sum_{i=1}^{N} x_i \ .$$

The associated standard uncertainty is the standard deviation of the sample mean,

$$u(x) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} (x - x_i)^2}$$

and the number of degrees of freedom is

$$\nu = N - 1 \ .$$

In the GUM, the values of x, $u(x)$ and ν characterise a measurement result and allow an uncertainty interval to be calculated.

3.2.2 Prefer rectangular coordinates when evaluating type-A uncertainty

If sample statistics are calculated from raw data expressed in polar coordinates potentially serious errors can occur.

1. The mean and standard deviation of data expressed in polar coordinates will, in general, differ from the same statistics evaluated from data expressed in rectangular coordinates, because of the non-linearity of the coordinate transformation.
2. Periodicity of phase values is difficult to handle correctly.

These concerns are discussed in detail in [7, 8].

3.3 Software

GTC includes a module `type_a` that has functions for evaluating type-A uncertainties. In this section, we show how samples of data can be processed to define an uncertain number representing the best estimate.

3.3.1 Type A with real data

The function `estimate` processes a sample of data and returns an uncertain number equal to the sample mean, with an appropriate standard uncertainty and degrees-of-freedom.

```
>>> lst = [1.5471269103260443, 0.70004668719095708, 1.1164742345100096,
    0.8356402220244048, 0.59867003630047511, 0.68451476827994484,
    0.67144840059703137, 1.124462023383666, 1.8840919720757041,
    0.77242428646855965, 1.1299361535758743]
>>> x = type_a.estimate(lst)
>>> x
ureal(1.0058941540666064, 0.12298722855464676, 10)
```

3.3.2 Type A with complex data

`estimate` can also be used with complex data. In this case, the uncertain complex number `z` is equal to the sample mean and has a covariance equal to the sample covariance divided by $\sqrt{N}$. The degrees of freedom is again $N - 1$.

```
>>> lst = [(2.7930733953935123-0.95702133611218265j),
    (-1.3963548361447962+1.8772611208988534j),
    (4.4095600996892959-0.31769375085812124j),
    (1.9696780062016057+1.5470577558103562j),
    (-1.5682672965610474-1.2630977607882938j),
    (-1.8173890585062633-2.0298147888216111j),
    (3.6666564314333998-0.64181830856510858j),
    (0.25620450093730829+3.4608431599179466j),
    (4.8593534859357437-1.6227300252491133j),
    (3.7355399848184181+1.3552598599089636j),
    (0.068958899466353474-0.234632126893902j)]
>>> z = type_a.estimate(lst)
>>> z
ucomplex( (1.5433648738785026+0.10669216356798064j),
    [0.5792536101218766,-0.03722844327254706,-0.03722844327254706,0.26762786531098554],
     10 )
```

4 Evaluating type B uncertainty

4.1 Main points

In RF and microwave measurements, a common source of type-B uncertainty arises when nothing is known about the phase of an influence quantity. Three cases are considered below, in which different information is available about the magnitude.

Another common problem arises when two influence quantities with unknown phase are multiplied. The methods described in §5.1 do not propagate an uncertainty for any product with zero-valued factors. So that case needs to be treated differently. A simple relation between the uncertainty of the product and the uncertainties of the factors is given below.

In this section, standard uncertainties in the real and imaginary components are always equal, so the component of uncertainty matrices (see §5.1), have a simple diagonal form

$$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} .$$

There is no need to write a matrix for each case, so only u is given.

4.1.1 Known magnitude

When the magnitude $|\mathbf{\Gamma}| = a$ is known, $\mathbf{\Gamma}$ could be attributed to any point on a circle around the origin, corresponding to a *uniform ring* of uncertainty.

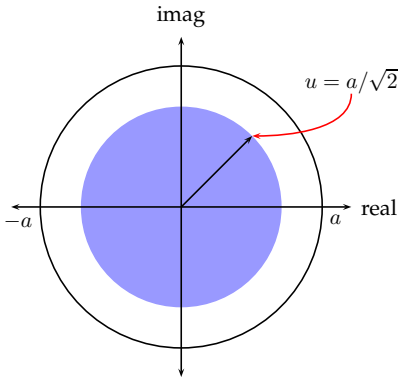

Figure 7: When the magnitude is known the complex value may lie anywhere on a ring of radius $a = |\mathbf{\Gamma}|$. The standard uncertainty $u(\mathbf{\Gamma}) = a/\sqrt{2}$.

The best estimate of $\mathbf{\Gamma}$ for the purpose of uncertainty calculation is zero. The standard uncertainty of the real and imaginary components is [5, 9]

$$u(\mathbf{\Gamma}) = \frac{a}{\sqrt{2}}$$

and the correlation coefficient $r = 0$.

4.1.2 Bounded magnitude

When an upper limit for the magnitude is known $|\mathbf{\Gamma}| \leq a$, $\mathbf{\Gamma}$ could be attributed to any point in a circular region around the origin (Figure 8). This corresponds to a *uniform*

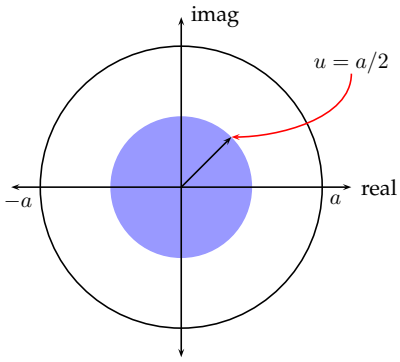

Figure 8: The uncertainty region is a disk when the magnitude is bounded above. The radius $a = |\mathbf{\Gamma}|$ and the standard uncertainty $u(\mathbf{\Gamma}) = a/2$.

disk of uncertainty. The standard uncertainty of the real and imaginary components is [5, 9]

$$u(\mathbf{\Gamma}) = \frac{a}{2}$$

and the correlation coefficient $r = 0$.

4.1.3 Magnitude estimate

When the magnitude is estimated $|\mathbf{\Gamma}| \approx a$, and the associated uncertainty is $u(a)$, $\mathbf{\Gamma}$ could be attributed to any point in a radially symmetric distribution around the origin.

The standard uncertainty in this case is [9]

$$u(\mathbf{\Gamma}) = \sqrt{\frac{a^2}{2} + u(a)^2}$$

and the correlation coefficient $r = 0$.

It is worth considering the limiting cases of this expression. When $a \gg u(a)$, the standard uncertainty reduces to a uniform ring and when $a \ll u(a)$ the standard uncertainty approaches the uncertainty of the measurement $u(a)$. The later situation is not uncommon. For example, the magnitude of the reflection coefficient of a precision load can be very close to zero.

Unlike the previous cases, the distribution associated with possible values of $\mathbf{\Gamma}$ does not have a simple geometry. However, most of its density is located close to the circle of radius a. The standard uncertainty of a uniform annulus, with internal radius $a - \sqrt{2}u(a)$ and external radius $a + \sqrt{2}u(a)$, yields the same standard uncertainty [5, 9, §2] and could be a helpful intuitive representation of this case (Figure 9).

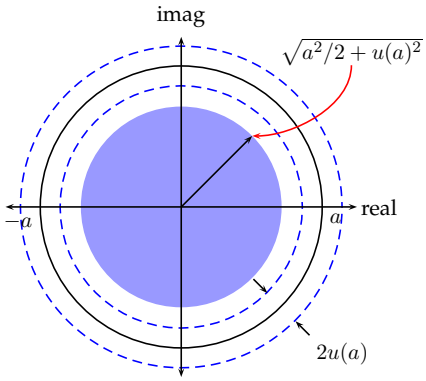

Figure 9: When the radius is estimated $a \approx |\mathbf{\Gamma}|$, with an uncertainty $u(a)$, the standard uncertainty $u(\mathbf{\Gamma}) = \sqrt{a^2/2 + u^2}$. The geometry of the uncertainty region can be represented intuitively as a uniform annulus.

4.1.4 Product of estimates

When a product of two complex quantities

$$\boldsymbol{G} = \mathbf{\Gamma}_1 \mathbf{\Gamma}_2$$

influences a measurement and the phases of both factors are unknown, the phase of $\boldsymbol{G} \approx 0$ will be unknown. This presents a problem for uncertainty calculation, because the conventional method will not propagate any uncertainty to the product.

In this special case, $\boldsymbol{G}$ can be treated as a single influence quantity with a value that could attributed to any point in radially symmetric distribution around the origin (and hence with an estimate of zero).

The standard uncertainty is

$$u(\boldsymbol{G}) = \sqrt{2} u(\mathbf{\Gamma}_1) u(\mathbf{\Gamma}_2)$$

and the correlation coefficient $r = 0$.

Note the standard method does propagate uncertainty to the product when the phase of one factor is known (but then only the uncertainty associated with the zero-valued factor is propagated). In such cases, consideration should be given to what is known about the factors and an appropriate type-B uncertainty should be chosen on that basis.

4.2 Additional comments

4.2.1 Type B uncertainty for real quantities

The GUM describes a number of type-B uncertainty distributions for real-valued estimates: uniform, triangular and arcsine (or U-shaped). They do not generally apply in the complex uncertainty problem. However, it is interesting to note that the arcsine distribution is the marginal distribution of a uniform ring distribution.

It is also important to note that real-valued quantities may influence the measurement of a complex quantity. So, univariate type-B distributions can appear in a complex uncertainty calculation. This does not pose a problem for the methods of uncertainty calculation described in Part 2 of this report.

4.2.2 These distributions are very different from Gaussian

The method of propagating uncertainty described in this report assumes that the combined error associated with measurement results has an approximately Gaussian distribution. The type-B distributions described in this section are not at all Gaussian-like, so it is important to be aware how much they influence the uncertainty of a measurement.

A recent study investigated the performance of uncertainty calculations in which type-B influences are involved [9]. It found that the coverage probability of uncertainty calculations remained satisfactory, provided the contribution from these influences is not dominant in the uncertainty budget.

4.2.3 Do these distributions exist?

The uniform ring and uniform disk distributions are instances of *a priori* distributions, based on information (or knowledge) available when preparing the uncertainty budget. They cannot be expected to describe the distribution of errors in a measurement procedure with a high degree of accuracy. They are used to provide a reasonable approximation of influence quantity errors in the absence of better information. If an error term is significant, it is always best to refine the evaluation of uncertainty.

The arcsine distribution (which is usually equivalent to the uniform ring) has been associated with mismatch uncertainty for many years [10]. However, a recent experimental study suggests that this practice has been conservative [11]. It is to be expected from the Central Limit Theorem that the net effect of many small errors will generate an approximately bivariate Gaussian error [12] and this now appears to be borne out in some RF practice [11].

4.2.4 Can degrees-of-freedom be associated with a type-B uncertainty?

The GUM provides an expression for a number of degrees-of-freedom when there is some uncertainty in a type-B uncertainty parameter [1, G.4.2]. A similar expression is applicable to the bivariate type-B distributions, see §6.2.3.

4.3 Software

4.3.1 Type-B distributions

GTC has functions that transform the parameter describing the extent of a type-B distribution (usually the radius) into a standard uncertainty.

```
>>> a, u_a = 1,0.1
>>> type_b.uniform_ring(a)
0.7071067811865475
>>> type_b.uniform_disk(a)
0.5
>>> type_b.uncertain_ring( (a,u_a) )
0.714142842854285
```

These can be used when defining uncertain complex numbers representing complex quantities with unknown phase.

```
>>> a, u_a = 1,0.1
>>> z_unknown_phi = ucomplex(0.0, type_b.uncertain_ring( (a,u_a) ) )
>>> z_unknown_phi
ucomplex( 0j, [0.51,0.0,0.0,0.51], inf )
>>> uncertainty(z_unknown_phi)
standard_uncertainty(real=0.714142842854285, imag=0.714142842854285)
```

4.3.2 Mismatch uncertainty

In a simple power measurement scenario[9], the measurement equation can be written as

$$P_{\mathrm{g}} = M P_{\mathrm{i}} \,, \tag{8}$$

where P_{i} is the net RF power available to a load with a reflection coefficient $\mathbf{\Gamma}_{\mathrm{s}}$, P_{g} is the power that the generator could deliver to an ideal load and

$$M = |1 - \mathbf{\Gamma}_{\mathrm{s}}\mathbf{\Gamma}_{\mathrm{g}}|^2 \tag{9}$$

is commonly referred to as the mismatch error. Mismatch depends on the complex reflection coefficients of the signal generator output, $\mathbf{\Gamma}_{\mathrm{g}}$, and $\mathbf{\Gamma}_{\mathrm{s}}$. Assuming that the phases of $\mathbf{\Gamma}_{\mathrm{g}}$ and $\mathbf{\Gamma}_{\mathrm{s}}$ are unknown, a type-B uncertainty should be associated with the estimate $M \approx 1$.

We are given the following information about the quantities, but there is no phase information for $\mathbf{\Gamma}_{\mathrm{s}}$ and $\mathbf{\Gamma}_{\mathrm{g}}$.

Quantity	Value	Uncertainty
$\lvert\mathbf{\Gamma}_{\mathrm{g}}\rvert$	< 0.310	-
$\lvert\mathbf{\Gamma}_{\mathrm{s}}\rvert$	< 0.083	-
P_i	$100\,\mu\mathrm{W}$	1%

To solve the problem, we begin by defining an uncertain number representing the product

$$\boldsymbol{G} = \mathbf{\Gamma}_{\mathrm{s}}\mathbf{\Gamma}_{\mathrm{g}}$$

```
>>> u_Gg = type_b.uniform_disk(0.310)
>>> u_Gs = type_b.uniform_disk(0.083)
>>> u_G = type_b.unknown_phase_product(u_Gg,u_Gs)
>>> G = ucomplex(0.0,u_G,label='GgGs')
>>> G
ucomplex( 0j, [8.27541125e-05,0.0,0.0,8.27541125e-05], inf, label=GgGs )
```

[9] See §8.1 of Part 1 of this report [2]

The generator power can be calculated directly for a particular incident power

```
>>> Pi = ureal(100E-6,1E-6,label='Pi')
>>> M = mag_squared(1.0-G)
>>> Pg = M * Pi
>>> summary(Pg)
'0.000100, u=2.1E-06, df=inf'
```

This result was obtained in §8.1 of Part 1 of this report [2], which also considered the more conventional assumption of uniform rings of uncertainty for $\mathbf{\Gamma}_\mathrm{s}$ and $\mathbf{\Gamma}_\mathrm{g}$.

GTC provides a convenient mapping `type_B.distribution` that can be used to change more easily between different type-B distributions. Here we show how the calculation could be made more flexible. Uniform rings are used, instead of uniform discs.

```
>>> type = 'uniform_ring'
>>> u_Gg = type_b.distribution[type](0.310)
>>> u_Gs = type_b.distribution[type](0.083)
>>> u_G = type_b.unknown_phase_product(u_Gg,u_Gs)
>>> G = ucomplex(0.0,u_G,label='GgGs')
>>> Pi = ureal(100E-6,1E-6,label='Pi')
>>> Pg = mag_squared(1.0-G) * Pi
>>> summary(Pg)
'0.000100, u=3.8E-06, df=inf'
```

We obtain $u(P_\mathrm{g}) = 3.8 \times 10^{-6}$ as expected [2].

5 Propagation of uncertainty

5.1 Main points

Propagation of uncertainty requires some analysis of the equation describing a measurement procedure (including data-processing). The measurement equation can be expressed as

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N) \ ,$$

where all quantities are complex-valued. Estimates of the inputs $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ are used to obtain the estimate $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N)$.

Uncertainty propagation require evaluation of the covariance matrix associated with $\boldsymbol{y}$

$$\mathbf{v}(\boldsymbol{y}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{u}_i(\boldsymbol{y}) \, \mathbf{r}_{ij} \, \mathbf{u}_j(\boldsymbol{y})' \ . \tag{10}$$

In this equation, the component of uncertainty matrices $\mathbf{u}_i(\boldsymbol{y})$ represent uncertainty contributions from each complex influence [13]

$$\begin{aligned} \mathbf{u}_i(\boldsymbol{y}) &= \left[\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}_i} \right] \mathbf{u}(\boldsymbol{x}_i) \ , \\ &= \begin{bmatrix} \frac{\partial Y_{\mathrm{re}}}{\partial X_{j\cdot\mathrm{re}}} & \frac{\partial Y_{\mathrm{re}}}{\partial X_{i\cdot\mathrm{im}}} \\ \frac{\partial Y_{\mathrm{im}}}{\partial X_{i\cdot\mathrm{re}}} & \frac{\partial Y_{\mathrm{im}}}{\partial X_{i\cdot\mathrm{im}}} \end{bmatrix} \begin{bmatrix} u(x_{i\cdot\mathrm{re}}) & 0 \\ 0 & u(x_{i\cdot\mathrm{im}}) \end{bmatrix} \end{aligned}$$

where $u(x_{i\cdot\mathrm{re}})$ and $u(x_{i\cdot\mathrm{im}})$ are standard uncertainties of the real and imaginary components of $\boldsymbol{x}_i$, respectively.

The correlation matrices $\mathbf{r}_{ij}$ in (10) contain the four correlation coefficients relating to the real and imaginary components of $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$

$$\mathbf{r}_{ij} = \begin{bmatrix} r_{i\cdot\mathrm{re},\, j\cdot\mathrm{re}} & r_{i\cdot\mathrm{re},\, j\cdot\mathrm{im}} \\ r_{i\cdot\mathrm{im},\, j\cdot\mathrm{re}} & r_{i\cdot\mathrm{im},\, j\cdot\mathrm{im}} \end{bmatrix} \ ,$$

when $i = j$ these matrices are symmetric

$$\mathbf{r}_{ii} = \begin{bmatrix} 1 & r_{i\cdot\mathrm{im},\, i\cdot\mathrm{re}} \\ r_{i\cdot\mathrm{im},\, i\cdot\mathrm{re}} & 1 \end{bmatrix} \ .$$

5.2 Additional comments

5.2.1 Propagation of uncertainty for real quantities

According to the *Law of Propagation of Uncertainty* (LPU) described in the GUM [1, §5], an equation among physical quantities defines a measurement procedure

$$Y = f(X_1, X_2, \cdots, X_N) \ ,$$

where all quantities are real-valued. Estimates of the influence quantities are used to estimate the measurand

$$y = f(x_1, x_2, \cdots, x_N) \ .$$

A standard uncertainty is associated with each estimate, denoted here as $u(x_i)$.

A component of uncertainty in y due to uncertainty in x_i can be defined as

$$u_i(y) = \frac{\partial y}{\partial x_i} u(x_i) \ ,$$

in which case the combined standard uncertainty associated with y is

$$u(y) = \left[\sum_{i=1}^{N} \sum_{j=1}^{N} u_i(y)\, r_{ij}\, u_j(y) \right]^{1/2} \ ,$$

where r_{ij} is the correlation coefficient between x_i and x_j.

5.2.2 GUM method applied to the complex problem

The covariance matrix for a complex quantity can be evaluated by several applications of the GUM LPU method for real quantities.

The measurement problem must be formulated in terms of a pair of measurands: the real and imaginary components of the complex quantity. A pair of functions describe these measurands in terms of $2N$ real-valued quantities (the real and imaginary components of the N complex influences)

$$\begin{aligned} Y_{\rm re} &= f_{\rm re}(X_1, X_2, \cdots, X_{2N}) \\ Y_{\rm im} &= f_{\rm im}(X_1, X_2, \cdots, X_{2N}) \ . \end{aligned}$$

The LPU should be applied separately to each measurement function, giving the standard variances

$$\begin{aligned} u(y_{\rm re})^2 &= \sum_{i=1}^{2N} \sum_{j=1}^{2N} u_i(y_{\rm re})\, r_{ij}\, u_j(y_{\rm re}) \\ u(y_{\rm im})^2 &= \sum_{i=1}^{2N} \sum_{j=1}^{2N} u_i(y_{\rm im})\, r_{ij}\, u_j(y_{\rm im}) \end{aligned}$$

and then the covariance between the real and imaginary components can be calculated as

$$u(y_{\rm re}, y_{\rm im}) = \sum_{i=1}^{2N} \sum_{j=1}^{2N} u_i(y_{\rm re})\, r_{ij}\, u_j(y_{\rm im}) \ .$$

5.2.3 An alternative matrix formulation

There is a concise matrix form of multivariate uncertainty propagation [14] that has been applied to RF and microwave measurements by Ridler and Salter [7, 8, 15].[10]

The N complex inputs to the measurement equation are represented by a $2N$-component column vector of real values (the real and imaginary components of the inputs)

$$\mathbf{x} = [x_{11}, x_{21}, x_{12}, x_{22}, \cdots, x_{1N}, x_{2N}]'$$

[10] The matrix method is also described in §6.2 of [3], several examples are given.

and the measurand estimate is a 2-element vector

$$\mathbf{y} = [y_1, y_2]' = \boldsymbol{f}(\boldsymbol{x})$$

The uncertainty associated with the inputs is represented by a $2N$-by-$2N$ covariance matrix $\mathbf{v}(\mathbf{x})$. The diagonal elements of $\mathbf{v}(\mathbf{x})$ are standard variances of the inputs and the off-diagonal terms are covariances.

The 2-by-2 covariance matrix uncertainty associated with the uncertainty of $\boldsymbol{y}$ is

$$\mathbf{v}(\mathbf{y}) = \left[\frac{\partial \mathbf{y}}{\partial \mathbf{x}'}\right] \mathbf{v}(\mathbf{x}) \left[\frac{\partial \mathbf{y}}{\partial \mathbf{x}'}\right]' ,$$

where $\left[\frac{\partial \mathbf{y}}{\partial \mathbf{x}'}\right]$ is a Jacobian matrix of partial derivatives

$$\left[\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}'}\right] = \begin{bmatrix} \frac{\partial y_1}{\partial x_{11}} & \frac{\partial y_1}{\partial x_{21}} & \frac{\partial y_1}{\partial x_{12}} & \frac{\partial y_1}{\partial x_{22}} & \cdots & \frac{\partial y_1}{\partial x_{1N}} & \frac{\partial y_1}{\partial x_{2N}} & \frac{\partial y_1}{\partial x_{1N}} & \frac{\partial y_1}{\partial x_{2N}} \\ \frac{\partial y_2}{\partial x_{11}} & \frac{\partial y_2}{\partial x_{21}} & \frac{\partial y_2}{\partial x_{12}} & \frac{\partial y_2}{\partial x_{22}} & \cdots & \frac{\partial y_2}{\partial x_{1N}} & \frac{\partial y_2}{\partial x_{2N}} & \frac{\partial y_2}{\partial x_{1N}} & \frac{\partial y_2}{\partial x_{2N}} \end{bmatrix} .$$

Note in this Jacobian matrix, each input is associated with a 2-by-2 submatrix, for example $\boldsymbol{x}_1$ is associated with

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_{11}} & \frac{\partial y_1}{\partial x_{21}} \\ \frac{\partial y_2}{\partial x_{11}} & \frac{\partial y_2}{\partial x_{21}} \end{bmatrix} .$$

These submatrices can be obtained from the complex partial derivatives of the measurement function using (11).

Implicit problems: Some problems in RF metrology are expressed implicitly

$$\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{y}) = 0 .$$

Propagation of uncertainty can again be carried out using matrix algebra (see [14]).

A case of particular interest is the calibration of a reflectometer using three calibration standards. Cox, *et al*, have described this [16][11] and it is discussed in Section 5 of [17]. Section 5.3.3 below also shows how this problem can be handled quite easily by software.

5.2.4 Simplifying matrix derivative calculations

When a measurement equation $\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N)$ is analytic, there is a way to reduce the number of partial derivatives that need to be evaluated for the $\mathbf{u}_i(\boldsymbol{y})$ component matrices.

Complex differentiation obtains a value $\boldsymbol{Z} = \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}_i}$ directly, which can then be transformed into a matrix by the relation

$$\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}_i} = \boldsymbol{Z} \leftrightarrow \begin{bmatrix} Z_{\mathrm{re}} & -Z_{\mathrm{im}} \\ Z_{\mathrm{im}} & Z_{\mathrm{re}} \end{bmatrix} . \tag{11}$$

This simple trick can significantly reduce the amount of calculus required.

[11] See also §6.3 of [3]

5.2.5 Real measurands with complex influence quantities

There are situations where a real-valued measurand is influenced by complex quantities (e.g., in the measurement of power, or attenuation).

In such cases, it is convenient to identify the intermediate results of the main measurement equation that involve complex quantities. Methods of uncertainty propagation for complex quantities can be applied to these intermediate problems. Then the results can be expressed in terms of real and imaginary components and the uncertainty analysis completed using the GUM LPU.

It is essential that all uncertainty components $u_j(\boldsymbol{y}_i)$, of the intermediate results $\boldsymbol{y}_i$, are correctly handled by the GUM LPU calculation, which may involve careful application of the chain rule for differentiation.[12]

5.2.6 Can component of uncertainty matrices be summarised as a magnitude?

The GUM defines a positive quantity called the *component of combined standard uncertainty* in y due to uncertainty in x_i

$$u_i(y) = \left| \frac{\partial y}{\partial x_i} \right| u(x_i) \ ,$$

which is useful for reporting the relative importance of influence quantities on the combined uncertainty of a result.

A single magnitude representing the four elements in $\mathbf{u}_i(\boldsymbol{y})$ matrices is also useful in this regard. When reporting the influence of a complex quantity in an uncertainty budget, the magnitude

$$\overline{u}_i(\boldsymbol{y}) = \sqrt{\frac{u_{i\cdot\mathrm{re}}(y_\mathrm{re})^2 + u_{i\cdot\mathrm{im}}(y_\mathrm{re})^2 + u_{i\cdot\mathrm{re}}(y_\mathrm{im})^2 + u_{i\cdot\mathrm{im}}(y_\mathrm{im})^2}{2}}$$

is a suitable summary value.

There is a relationship between $\overline{v}(\boldsymbol{y})$ (the mean variance, defined in the §2.2.6) and $\overline{u}_i(\boldsymbol{y})$. When all influence estimates are independent (including the real and imaginary components) then

$$\overline{v}(\boldsymbol{y}) = \sum_i \overline{u}_i(\boldsymbol{y})^2 \ ,$$

where the sum is over all the influences of $\boldsymbol{y}$.

$\overline{u}_i(\boldsymbol{y})$ is equivalent to the expression for a component of uncertainty defined in Part 1 of this report [2, §6.1 eq 7], when the assumptions of Part 1 hold.

5.3 Software

GTC software streamlines the calculation of uncertainty in real and complex valued problems. It automates the propagation of uncertainty, and the calculation of degrees-of-freedom, which enables relatively complicated problems to be solved numerically.

[12] One of the advantages of software such as GTC is that this process is automatic.

Later in this section, we show how the calibration of one port of a vector network analyser can be handled. The example is simplified, but still rather longer than other software examples in this report. It is hoped that the essential features of the approach to calculation can be appreciated.

The first example involves a calculation of equivalent reflection coefficient. We will use this to illustrate the use of functions and object-oriented programming techniques, which will be needed in the VNA example.

5.3.1 Equivalent reflection coefficient

There is a well-known expression for the reflection coefficient looking in to one port of a linear passive two-port network when the other is terminated with a reflection coefficient $\mathbf{\Gamma}$,

$$\mathbf{\Gamma}' = \boldsymbol{S}_{11} + \frac{\boldsymbol{S}_{21}\boldsymbol{S}_{12}\mathbf{\Gamma}}{1 - \boldsymbol{S}_{22}\mathbf{\Gamma}} \,.$$

If the values and uncertainty of the S-parameters and $\mathbf{\Gamma}$ are known, we can easily calculate $\mathbf{\Gamma}'$. For example

```
>>> S11 = ucomplex( 0.05 - 0.03j, 0.02, label='S11')
>>> S21 = ucomplex( 0.91 - 0.06j, 0.03, label='S21')
>>> S12 = ucomplex( 0.95 - 0.02j, 0.03, label='S12')
>>> S22 = ucomplex( 0.07 + 0.02j, 0.02, label='S22')

>>> Gamma = ucomplex( 0.3+0.2j, (0.02,0.04), label = 'Gamma')
```

the effective reflection coefficient is

```
>>> G_eff = S11 + (S12 * S21 * Gamma)/ (1 - S22 * Gamma)
>>> value(G_eff)
(0.32554827871218145+0.12830210129628042j)
>>> uncertainty(G_eff)
standard_uncertainty(real=0.03064091238057904, imag=0.0435781566142131)
```

Of course, it may be convenient to define a function that calculates the effective reflection coefficient. This would be preferable if the expression were used for a number of different sets of measurements.

A practical implementation is

```
>>> def gamma_equiv(Gamma,S11,S21,S12,S22):
...  """Return the effective reflection coefficient"""
...  num = S12 * S21 * Gamma
...  den = 1.0 - S22 * Gamma
...  return S11 + num/den
...
```

To call the function, print the result and then generate an uncertainty budget, we write

```
>>> Gamma_prime = gamma_equiv(Gamma,S11,S21,S12,S22)
>>> print summary(Gamma_prime)
```

```
>>> for influence,u_cpt in reporting.budget(Gamma_prime):
... print influence,':',u_cpt
```

The output is (the numbers displayed are the summary values $\overline{\mathbf{u}}_i(\boldsymbol{y})$)

```
(0.326+0.128j), u=[0.031,0.044], r=0.033, df=inf
Gamma : 0.0283476068202
S11 : 0.02
S21 : 0.0104536840276
S12 : 0.0100330480747
S22 : 0.00233071809794
```

A more sophisticated solution, that might be made more convenient when the reflection coefficient is to be calculated repeatedly, will define a class to hold S-parameters.

```
class TwoPort(object):
    def __init__(self,S11,S21,S12,S22):
        self.S11 = S11
        self.S12 = S12
        self.S21 = S21
        self.S22 = S22
    def gamma_equiv(self,gamma):
        num = self.S12 * self.S21 * gamma
        den = 1 - self.S22 * gamma
        return self.S11 + num/den
```

Objects of this class store S-parameters and can calculate the effective reflection coefficient given a value for $\boldsymbol{\Gamma}$

```
>>> two_port = TwoPort(S11,S12,S21,S22)
>>> G_eff = two_port.gamma_equiv(Gamma)
>>> value(G_eff)
(0.32554827871218145+0.12830210129628042j)
>>> uncertainty(G_eff)
standard_uncertainty(real=0.03064091238057904, imag=0.0435781566142131)

>>> Gamma2 = ucomplex( 0.9+0.12j, (0.03,0.03), label = 'Gamma2')
>>> G_eff2 = two_port.gamma_equiv(Gamma2)
>>> print summary(G_eff2)
(0.885+0.032j), u=[0.055,0.055], r=0.0, df=inf
```

5.3.2 One port VNA measurement

The conventional error model for one port of a vector network analyser is shown in Fig 10. The error parameters are shown along the red branches of the flow graph and a device with reflection coefficient $\boldsymbol{\Gamma}$ is connected to the measurement port. Raw measurement data is obtained as the ratio of the complex amplitudes

$$\gamma_{\mathrm{raw}} = \frac{\boldsymbol{b}}{\boldsymbol{a}} \,.$$

VNA calibration is a process that estimates $\boldsymbol{E}_{\mathrm{D}}$, $\boldsymbol{E}_{\mathrm{S}}$ and $\boldsymbol{E}_{\mathrm{R}}$. Then, in use, these values are used to correct raw data to obtain an estimate of the measured reflection

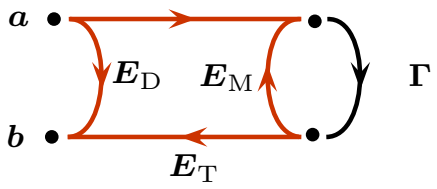

Figure 10: One-port VNA error model.

coefficient

$$\gamma = \frac{\gamma_{raw} - e_D}{e_S(\gamma_{raw} - e_D) - e_R} .$$

For this example, we ignore other sources of measurement error, such as noise, connector and cable effects, etc, and show only how uncertainty in error parameters can be handled.

We implement the correction using a class to represent the calibrated VNA. This conveniently represents the fact that a VNA is calibrated once and retains a set of parameters (with unchanging errors) until it is calibrated again.

We define the class as follows[13]

```
class VNA(object):
    """
    Represents a calibrated VNA port
    """
    def __init__(self,e_D,e_S,e_R):
        """Class constructor
        """
        self.e_D = e_D
        self.e_S = e_S
        self.e_R = e_R

    def gamma(self, gamma_raw):
        """Return an estimate of gamma
        """
        num = gamma_raw - self.e_D
        den = self.e_S * num + self.e_R

        return num/den
```

We suppose that error-parameter data is available at a particular frequency.

```
>>> e_D = ucomplex(
...  0.01 - 0.005j,
...  [0.005, 0.007],
...  label = 'e_D'
... )
>>> e_S = ucomplex(
...  -0.01 + 0.015j,
...  [0.006, 0.011],
```

[13] Readers familiar with object-oriented programming will appreciate that the VNA class could be augmented with methods to implement a more complete instrumentation error model. In this way a better solution can be developed. Furthermore, a collection of VNA objects could be used to correct raw measurements at different frequencies.

```
...  label = 'e_S'
... )
>>> e_R = ucomplex(
...  0.91 + 0.07j,
...  [0.004, 0.007],
...  label = 'e_R'
... )
```

An instance of the VNA class can correct raw data at this frequency.

```
>>> vna = VNA(e_D,e_S,e_R)

>>> gamma_raw = 0.03-0.01j
>>> gamma = vna.gamma(gamma_raw)
>>> print summary(gamma)
```

The summary string is

```
(0.0214-0.0072j), u=[0.0055,0.0077], r=0.053, df=inf
```

The uncertainty components associated with the three error parameters can be examined. The following lines print out a list, in order of the component with the greatest magnitude (the greatest $\overline{\mathbf{u}}_i(\boldsymbol{\gamma})$, see §5.2.6).

```
>>> for l,u in reporting.budget(gamma):
...  print "%s: %G" % (l,u)
...
```

The two significant components are[14]

```
e_D: 0.00666609
e_R: 0.000141119
```

If a highly reflecting device were measured, the uncertainty budget changes. For example,

```
>>> gamma_raw = 0.93-0.01j
>>> gamma = vna.gamma(gamma_raw)
>>> print summary(gamma)
(1.0119-0.10j), u=[0.0098,0.016], r=0.19, df=inf

>>> for l,u in reporting.budget(gamma):
...  print "%s: %G" % (l,u)
...
```

Displays the budget

[14] By default, `reporting.budget` filters out very small components (here the component associated with $\boldsymbol{E}_{\mathrm{S}}$ is about 4×10^{-6}). However, by writing `trim=0` after the argument `gamma` in the call to `budget`, all components will be displayed.

```
e_S: 0.00916111
e_D: 0.00678185
e_R: 0.00640709
```

Whereas the directivity $\boldsymbol{e}_{\mathrm{D}}$ dominated for a weakly reflecting device, the source match $\boldsymbol{e}_{\mathrm{S}}$ is now the most important component (it was insignificant in the first case).

The full matrix representation of the uncertainty can also be obtained using the function `reporting.u_component`

```
>>> for influence in [e_D,e_S,e_R]:
...  print influence.label,":",reporting.u_component(gamma,influence)
...
```

This produces

```
e_D : u_components(rr=-0.005541655794809207, ri=-0.000847810174916314, ir
    =0.0006055786963687957, ii=-0.007758318112732889)
e_S : u_components(rr=-0.006084523249288161, ri=-0.0022204537755146805, ir
    =0.0012111566048261895, ii=-0.01115495929036163)
e_R : u_components(rr=-0.00441371428645518, ri=-0.0014939087379971792, ir
    =0.000853662135998388, ii=-0.007724000001296565)
```

The two-letter labels in the display identify matrix elements. For example, `rr` is the partial derivative of the real component of γ with respect to the real component of the influence quantity, `ir` is the partial derivative of the imaginary component of γ with respect to the real component of the influence quantity, etc.

5.3.3 VNA calibration

Calibration of one port of a VNA involves measuring three standards, each of which has a given value (perhaps from a calibration report). The three pairs of measured and given values are used to calculate three error-correction terms, which can then be used to adjust raw VNA readings for systematic errors during VNA operation.

The correction terms can be found by solving a set of simultaneous equations

$$\begin{bmatrix} \boldsymbol{\Gamma}_1 & 1 & -\boldsymbol{\Gamma}_1^{\mathrm{m}}\boldsymbol{\Gamma}_1 \\ \boldsymbol{\Gamma}_2 & 1 & -\boldsymbol{\Gamma}_2^{\mathrm{m}}\boldsymbol{\Gamma}_2 \\ \boldsymbol{\Gamma}_3 & 1 & -\boldsymbol{\Gamma}_3^{\mathrm{m}}\boldsymbol{\Gamma}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}_1^{\mathrm{m}} \\ \boldsymbol{\Gamma}_2^{\mathrm{m}} \\ \boldsymbol{\Gamma}_3^{\mathrm{m}} \end{bmatrix}$$

where $\boldsymbol{\Gamma}_1$, etc, are the given reflection coefficients of the standards and $\boldsymbol{\Gamma}_1^{\mathrm{m}}$, etc, are the corresponding measured values.

This can be solved by standard methods of linear algebra. Then, in one final step, the conventional VNA error terms are obtained from $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$

$$\begin{aligned} \boldsymbol{e}_{\mathrm{D}} &= \boldsymbol{B} \ , \\ \boldsymbol{e}_{\mathrm{S}} &= -\boldsymbol{C} \ , \\ \boldsymbol{e}_{\mathrm{R}} &= \boldsymbol{A} - \boldsymbol{B}\boldsymbol{C} \ . \end{aligned}$$

The software implementation for this problem uses a pre-defined function to solve the simultaneous equations above and return the error parameters.[15] A suitable function is

```
#-----------------------------------------------------------------------
def osl(measured,nominal):
    """
    Return the 1-port errors in a sequence (E_D, E_S, E_R)

    'measured', 'nominal' are 3-element sequences of complex numbers

    The function solves a system of 3 simultaneous equations
    to obtain the three complex reflectometer errors.

    """
    # H = [ (nominal[0], unity, -nominal[0] * measured[0]),
    # etc
    # ]
    H = la.array( [ (n,1.0,-n * m) for m,n in zip(measured,nominal) ] )
    b = la.array( measured )

    ABC = la.solve( H,b )

    E_D=ABC[1]
    E_S=-ABC[2]
    E_R=ABC[0] - ABC[1] * ABC[2]

    return (E_D,E_S,E_R)
```

The following sequences contain uncertain numbers representing raw VNA data and given values for the open, short and load standards used.[16]

```
>>> measured = (
...  ucomplex(-0.188 - 0.902j,0.05), # std 1
...  ucomplex(0.239 + 0.936j,0.05), # std 2
...  ucomplex(0.006 + 0.007j,0.05), # std 3
... )

>>> nominal = (
...  ucomplex(-1 + 0j,0.01),  # std 1
...  ucomplex(1 + 0j,0.01),  # std 2
...  ucomplex(0 + 0j,0.01),  # std 3
... )
```

The VNA error port parameters can now be obtained directly

```
errors = osl(measured,nominal)

print "One-port errors"
print "---------------"
print 'e_D', summary( errors[0] )
```

[15] This example is also described in the GTC documentation.

[16] Again, for simplicity, we do not envisage a detailed uncertainty budget for the measurement or the calibration standards used in this example. However, the reader should keep in mind that *any* uncertain complex numbers may be used in the sequences of measured and nominal values. In practice, the measured values would be uncertain numbers with a number of influences, from such things as connector errors and instrumentation noise. The given values are likely to be taken from a calibration certificate data.

```
print 'e_S', summary( errors[1] )
print 'e_R', summary( errors[2] )
```

which displays

```
One-port errors
---------------
e_D (0.0060+0.0070j), u=[0.051,0.051], r=0.00, df=inf
e_S (0.015-0.018j), u=[0.066,0.066], r=0.00, df=inf
e_R (0.213+0.919j), u=[0.036,0.036], r=0.00, df=inf
```

Correlation between the uncertainty components associated with different error terms can also be examined. The most significant correlation, in this case, arises between the directivity and the source-match. The following code displays individual correlation coefficients

```
q = get_correlation(errors[0],errors[1])

print "Correlation between components of e_S and e_D"
print "---------------------------------------------"
print " e_S_re and e_D_re :" , q.rr
print " e_S_re and e_D_im :" , q.ri
print " e_S_im and e_D_re :" , q.ir
print " e_S_im and e_D_im :" , q.ii
```

The output is

```
Correlation between components of E_S and E_D
---------------------------------------------
  e_S_re and e_D_re : -0.184749680584
  e_S_re and e_D_im : 0.795245697689
  e_S_im and e_D_re : -0.795245697689
  e_S_im and e_D_im : -0.184749680584
```

6 Degrees of freedom

6.1 Main points

Following §5.1, a function of physical quantities describes a measurement procedure

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N) \ .$$

A standard uncertainty matrix

$$\mathbf{u}(\boldsymbol{y}) = \begin{bmatrix} u(y_{\mathrm{re}}) & 0 \\ 0 & u(y_{\mathrm{im}}) \end{bmatrix}$$

is associated with the measurand estimate

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N) \ ,$$

where $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ estimate $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N$. Each input $\boldsymbol{x}_i$ also has an associated standard uncertainty matrix $\mathbf{u}(\boldsymbol{x}_i)$ and a number of degrees of freedom ν_i, which may be infinite.

A number of effective degrees of freedom ν_{eff} can be associated with the uncertainty of $\boldsymbol{y}$ when the input estimates $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ are independent [18].[17] [18]

Defining the set of 2×2 matrices

$$\mathbf{v}_i = \begin{bmatrix} v_{i\cdot 11} & v_{i\cdot 12} \\ v_{i\cdot 21} & v_{i\cdot 22} \end{bmatrix} = \mathbf{u}_i(\boldsymbol{y})\, \mathbf{r}_{ii}\, \mathbf{u}_i(\boldsymbol{y})' \ ,$$

we calculate

$$A = 2\left(\sum v_{i\cdot 11}\right)^2 = 2v_{11}^2$$
$$D = \sum v_{i\cdot 11} \sum v_{i\cdot 22} + \left(\sum v_{i\cdot 12}\right)^2 = v_{11}v_{22} + v_{12}v_{21}$$
$$F = 2\left(\sum v_{i\cdot 22}\right)^2 = 2v_{22}^2$$

then

$$a = 2\sum \frac{v_{i\cdot 11}^2}{\nu_i}$$
$$d = \sum \frac{v_{i\cdot 11}v_{i\cdot 22} + v_{i\cdot 12}^2}{\nu_i}$$
$$f = 2\sum \frac{v_{i\cdot 22}^2}{\nu_i}$$

and, finally, effective degrees of freedom is

$$\nu_{\mathrm{eff}} = \frac{A + D + F}{a + d + f} \ . \tag{12}$$

Note that ν_{eff} is associated with the measurement of a *complex quantity* and is required to calculate the coverage factor $k_{p,2}$ for an uncertainty region (see §2.1).

[17] Correlation is permitted between the real and imaginary components of each input quantity with finite degrees of freedom, but not between the components of different quantities unless they have infinite degrees of freedom.

[18] There is an important exception to the restriction on correlation when a set of estimates are considered to be samples from a multivariate distribution [19]. In that case, each estimate obtained has the same number of degrees of freedom and the method of calculating an effective degrees of freedom can be modified [19]. GTC software applies this modified calculation when a group of estimates are identified.

6.2 Additional comments

6.2.1 The Welch-Satterthwaite formula for real quantities

The effective degrees of freedom for real-valued quantities is calculated by the Welch-Satterthwaite formula (WS) in the GUM [1, G.4]. The effective degrees of freedom are used to calculate the coverage factor for an uncertainty interval.

When a measurement is described by a function of physical quantities

$$Y = f(X_1, X_2, \cdots, X_N) \ ,$$

a standard uncertainty $u(y)$ is associated with the estimate of the measurand

$$y = f(x_1, x_2, \cdots, x_N) \ .$$

There is a standard uncertainty $u(x_i)$ associated with each of the influence quantity estimates and also a number of degrees of freedom ν_i, which may be infinite.

The WS formula is used to find the effective degrees of freedom associated with $u(y)$

$$\frac{u(y)^4}{\nu_{\text{eff}}} = \sum_{i=1}^{N} \frac{u_i^4(y)}{\nu_i} \ . \tag{13}$$

It is stipulated in the GUM that the WS formula is only valid when all quantity estimates with finite degrees of freedom are uncorrelated. However, this restriction can be relaxed in certain cases [19, 20]. In particular, if a group, or groups, of input quantities are estimated from a set of simultaneous observations (effectively sampling from a multivariate distribution), then the group, or groups, can be combined to obtain a single component of uncertainty with $\nu = n - 1$ degrees of freedom. This component, or components, can be used in the WS formula to evaluate the effective degrees of freedom.

The relaxation of the GUM restriction on correlation is helpful. Many RF measurements have a real-valued measurand (e.g., power or attenuation) that is influenced by complex quantities. When there is a small number of repeat measurements of complex influence quantities (type-A evaluation), there is usually correlation between the estimates of real and imaginary components. This correlation can be handled as described above.

Note, however, that this approach is not applicable when correlation between estimates has been *caused* by functional dependence (see [19]), as is the case, for instance, for the error-parameters of a VNA.

6.2.2 Are degrees-of-freedom important?

Degrees-of-freedom is a measure of the size of the sample used to estimate a quantity.

When the degrees-of-freedom is small, the standard uncertainty obtained from a sample of data will be variable and may on occasions be much smaller than the standard deviation of the underlying errors. So the area of the uncertainty region must be increased to provide the required coverage probability (see §2.1).

A quick look at the table below (the same as Table 1, on page 7) shows that the coverage factor $k_{2,p}$ is always larger than k_p for the same degrees of freedom. As a

consequence, the degrees-of-freedom is arguably *more* important for complex measurements than it is for real-valued quantities.

ν	$k_{2,0.95}$	$k_{0.95}$	ν	$k_{2,0.95}$	$k_{0.95}$
2	28.3	4.3	7	3.5	2.4
3	7.6	3.2	8	3.3	2.3
4	5.1	2.8	9	3.2	2.3
5	4.2	2.6	10	3.1	2.2
6	3.7	2.5	50	2.6	2.0
			∞	2.45	1.96

What is the effect of ignoring the degrees of freedom completely? If an uncertainty statement is evaluated using the coverage factor for infinite degrees of freedom, when in fact there are finite degrees of freedom, then the actual coverage probability will be less then nominal.

For example, Table 2 shows how actual coverage probability drops when the coverage factors $k_p = 1.96$ and $k_{2,p} = 2.45$ are used to evaluate 95% expanded uncertainty intervals and regions instead of correct values for three sample sizes ($\nu = 3, 5, 10$).

Table 2: Coverage probability of uncertainty statements evaluated ignoring finite degrees of freedom

ν	coverage (real)	coverage (complex)
3	85%	66%
5	89%	78%
10	92%	88%

Clearly, it is important to take the degrees of freedom into account when calculating a coverage factor. It is *even more* important to do so when calculating complex uncertainty regions. Unfortunately, the notion of degrees-of-freedom seems to have fallen from favour in recent years. It should not be overlooked.[19]

6.2.3 Effective degrees-of-freedom for type-B uncertainties

An expression in the GUM can be used to evaluate a number of effective degrees-of-freedom when there is uncertainty about the exact width of the distribution associated with a type-B uncertainty.

An analogous expression applies to case of a radially symmetric type-B distribution centered on the origin (the unknown phase problem). In that case [9]

$$\nu_{\text{eff}} \approx \frac{1}{2}\left[\frac{u(\boldsymbol{x}_i)}{\Delta u(\boldsymbol{x}_i)}\right]^2 ,$$

which turns out to be the same expression used with GUM uncertainties [1, G.4.2]. For example, if an estimate of $|\boldsymbol{\Gamma}|$ is considered to be reliable to about 10%, i.e. $|\boldsymbol{\Gamma}|/\Delta|\boldsymbol{\Gamma}| \approx 10$, then $\nu_{\text{eff}} = 50$ could be associated with the standard uncertainty $u(\boldsymbol{\Gamma})$.

[19] Degrees of freedom is not mentioned in reference [21].

6.3 Software

GTC software performs degrees of freedom calculations automatically, which can greatly simplify uncertainty calculations.

6.3.1 Calculating effective degrees-of-freedom

This example shows how the degrees of freedom associated with the sum of two complex estimates is available as an attribute of the sum.

```
>>> z1 = ucomplex(1-3j,(.1,0.2,0.2,.6),12,label='z1')
>>> z2 = ucomplex(1.54-0.3j,(.01,1.2),7,label='z2')
>>> z = z1 + z2
>>> z
ucomplex( (2.54-3.3j),
    [0.10010000000000001,0.20000000000000004,0.20000000000000004,2.04],
    12.962615643943534 )
>>> z.df
12.962615643943534
```

Note that the effective degrees of freedom calculation changes automatically to the Welch-Satterthwaite formula if an uncertain real number is involved.

```
>>> summary( magnitude(z) )
'4.2, u=1.1, df=10.5'
```

Also note that this calculation does not violate the restriction on correlation between influence quantity estimates. Although the definition of `z1` does have a non-zero covariance term (and hence there is correlation between the real and imaginary components), the components are considered to come from a multivariate sample, in the sense described in §6.2.1, and so the degrees of freedom can be calculated.

6.3.2 Calculating expanded uncertainties for correlated inputs

An example in Appendix H.2 of the GUM illustrates the case of correlated input estimates in a measurement problem [1, H2].

Five observations each of three circuit parameters are made in an electrical circuit: the voltage across the terminals of a component, the current passing through the component and the phase difference between the current and voltage.

The measurement objective is to estimate the impedance of the circuit element, so if v, i and ϕ are the circuit parameters, then the measurand

$$z = \frac{v}{i}\, e^{\mathrm{j}\phi} \ .$$

The values and standard uncertainties of the input parameters can be obtained by simple analysis of the sampled data. However, there is correlation between the estimates. Effectively the data sample for v, i and ϕ comes from a tri-variate distribution, so we are entitled to treat the data as a special case and proceed with the degrees of freedom calculation (see §4.1 and §4.1.1 in [20]).

The following code takes the raw data, obtains uncertain real numbers based on sample statistics, including correlations, and obtains an uncertain number for the resistance

```
>>> V = [5.007,4.994,5.005,4.990,4.999]
>>> I = [19.663E-3,19.639E-3,19.640E-3,19.685E-3,19.678E-3]
>>> phi = [1.0456,1.0438,1.0468,1.0428,1.0433]
>>> v,i,p = type_a.multi_estimate_real((V,I,phi),labels=('V','I','phi'))
>>> v
ureal(4.999, 0.003209361307176l794, 4, label=V)
>>> i
ureal(0.019661, 9.471008394041335e-06, 4, label=I)
>>> p
ureal(1.04446, 0.0007520638270785368, 4, label=phi)

>>> r = v/i*cos(p)
>>> r
ureal(127.73216992810208, 0.07107140739699543, 4.0)
>>> reporting.uncertainty_interval(r)
expanded_uncertainty(lower=127.53484406624997, upper=127.92949578995419)
```

Here `uncertainty_interval` finds the uncertainty for a 95% level of confidence for the component's resistance.

Alternatively, had the raw input data been defined as complex numbers, the problem could be handled using uncertain complex numbers

```
>>> I = [ complex(x) for x in (19.663E-3,19.639E-3,19.640E-3,19.685E
    -3,19.678E-3) ]
>>> V = [ complex(x) for x in (5.007,4.994,5.005,4.990,4.999)]
>>> P = [ complex(0,p) for p in (1.0456,1.0438,1.0468,1.0428,1.0433) ]

>>> v,i,p = type_a.multi_estimate_complex( (V,I,P) )

>>> get_correlation(v.real,i.real)
-0.355311219817512

>>> z = v/i*exp(p)
>>> z.real
ureal(127.73216992810208, 0.0710714073969954, 4.0)
>>> get_correlation(z.real,z.imag)
-0.5884297844235157

>>> reporting.uncertainty_interval(z.real)
expanded_uncertainty(lower=127.53484406624997, upper=127.92949578995419)
```

7 Concluding comments

7.1 The second GUM Supplement

During the final stages of preparing this report, a GUM-supplement (Supplement) dealing with multivariate measurands was released [3].

The Supplement offers guidance for the general problem of evaluating measurement uncertainty when the measurand consists of more than one real-valued quantity. Complex measurands are therefore within the range of the Supplement, but are not its sole focus.

There is relatively little overlap between the material in the Supplement and this report, but footnotes have been added to refer to appropriate sections in the Supplement.

This report deals with the evaluation of measurement uncertainty for complex-valued quantities in more detail than the Supplement is is focused on RF and microwave measurement problems. For instance, it describes how to evaluate uncertainty when some estimates have been made from small numbers of observations (finite degrees-of-freedom), which is not mentioned in the Supplement. It also describes how to handle the type-B uncertainties associated with the unknown-phase problem, which is not considered in the Supplement either.

The method of uncertainty propagation described in §5.1 is a generalisation of the familiar GUM method of uncertainty propagation. We believe that this presentation is useful to the metrologist, who will be thinking in terms of the individual complex-valued influence quantities. The Supplement gives a more general matrix formulation of uncertainty propagation that will be preferred no doubt by mathematicians. Nevertheless, the different ways of presenting formulae for propagating uncertainty here and in the Supplement[20] are equivalent for complex-valued problems.

7.1.1 Propagation of distributions

The Supplement also describes a method of evaluating uncertainty that might best be be called 'propagation of distributions' (PoD). It is implemented in the Supplement using a Monte Carlo method and so is often referred as the 'Monte Carlo method'.

PoD takes a fundamentally different view of the uncertainty problem. It involves the notion of a 'state of knowledge' distribution representing measurands and influence quantities. This is a notion that is easily confused with Monte Carlo simulation of physical systems, and hence measurement errors. However, the intuition that a scientist or engineer may have for simulations of physical systems, does not carry over to the statistical problem of PoD. We feel that this point needs emphasis. It is mentioned briefly in the Supplement (see NOTE 2 to §5.6.1), but that brief footnote is easily overlooked.

The Supplement asserts in that (§1 [3])

[20] more precisely, the 'GUM uncertainty framework' described in §6 of [3].

Appropriate use of the Monte Carlo method would be expected to provide valid results when the applicability of the GUM uncertainty framework is questionable.

However, there is evidence that this claim is unjustified. Indeed, on the basis of several investigations, it appears that the GUM uncertainty framework is more robust than the Monte Carlo method. A discussion is found in [22, 23] and the references therein, as well as evidence that PoD does not always produce satisfactory uncertainty statements in [24, 25].

Acknowledgement

This work was funded by the New Zealand Government as part of a contract for the provision of national measurement standards. The author is grateful to Dr. Robin Willink for many useful discussions that have contributed to this document.

References

[1] BIPM, IEC, IFCC, ISO, IUPAC, IUPAP, and OIML. *Evaluation of measurement data Guide to the expression of uncertainty in measurement JCGM 100:2008 (GUM 1995 with minor corrections)*. BIPM Joint Committee for Guides in Metrology, Paris, Sèvres, 1st edition, 2008. (available on-line here). 1, 16, 21, 24, 36, 37, 38

[2] B D Hall. Notes on complex measurement uncertainty – part 1. Industrial Research Report 2483, Industrial Research Limited, Measurement Standards Laboratory, November 2010. (text available here). 1, 7, 22, 23, 27

[3] BIPM, IEC, IFCC, ISO, IUPAC, IUPAP, and OIML. *Evaluation of measurement data Supplement 2 to the "Guide to the expression of uncertainty in measurement" – Extension to any number of output quantities JCGM 102:2011*. BIPM Joint Committee for Guides in Metrology, Paris, Sèvres, 1st edition, 2011. 3, 4, 6, 25, 26, 40

[4] Richard A Johnson and Dean A Wichern. *Applied Multivariate Statistical Analysis*. Prentice Hall, 1998. 4, 6

[5] B D Hall. Some considerations related to the evaluation of measurement uncertainty for complex-valued quantities in radio frequency measurements. *Metrologia*, 44:L62–L67, 2007. (text available here). 8, 18, 19

[6] D F Williams, C M Wang, and U Arz. In-Phase/Quadrature Covariance-Matrix Representation of the Uncertainty of Vectors and Complex Numbers. *ARFTG Conf. Dig.*, 68, Dec 2006. 8

[7] N M Ridler and M J Salter. An approach to the treatment of uncertainty in complex S-parameter measurements. *Metrologia*, 39:295–302, 2002. 16, 25

[8] N M Ridler and M J Salter. Evaluating and expressing uncertainty in complex S-parameter measurements. *ARFTG Conf. Dig.*, 56:63–75, 2000. 16, 25

[9] B D Hall. On the expression of measurement uncertainty for complex quantities with unknown phase. *Metrologia*, 48:324–332, 2011. (text available here). 18, 19, 21, 37

[10] I A Harris and F L Warner. Re-examination of mismatch uncertainty when measuring microwave power and attenuation. *IEE Proceedings H: Microwaves, Optics and Antennas*, 128(1):35 – 41, 1981. 21

[11] M Dobbert and J Gorin. Revisiting mismatch uncertainty with the Rayleigh Distribution. In *NCSL International Workshop and Symposium*, Washington, D.C., USA, August 25 2011. NCSLI. 21

[12] J A Mullen and W L Prichard. The statistical prediction of Voltage Standing-wave ratio. *IRE Transactions Microwave Theory and Techniques*, pages 127–130, 1957. 21

[13] B D Hall. On the propagation of uncertainty in complex-valued quantities. *Metrologia*, 41(3):173–177, 2004. (text available here). 24

[14] K Weise. Treatment of uncertainties in precision measurements. *IEEE Trans. Instrum. Meas.*, IM-36:642–645, 1987. 25, 26

[15] N M Ridler and M J Salter. Propagating S-parameter uncertainties to other measurement quantities. *ARFTG Conf. Dig.*, 58, 2001. 25

[16] M G Cox, M P Dainton, P M Harris, N M Ridler, and P R Young. A generalised treatment of the uncertainty in calibration and measurement of vector-indicating microwave reflectometers. CETM 10, National Physical Laboratory, September 1999. 26

[17] B D Hall. Calibration uncertainty for a 3-sampler vector network analyser. Technical Report 1110, Industrial Research Limited, 2006. (text available here). 26

[18] R Willink and B D Hall. A classical method for uncertainty analysis with multidimensional data. *Metrologia*, 39:361–369, 2002. 35

[19] R Willink. An inconsistency in uncertainty analysis relating to effective degrees of freedom. *Metrologia*, 45:63–67, 2008. 35, 36

[20] R Willink. A generalization of the Welch-Satterthwaite formula for use with correlated uncertainty components. *Metrologia*, 44:340–349, 2007. 36, 38

[21] BIPM, IEC, IFCC, ISO, IUPAC, IUPAP, and OIML. *Evaluation of measurement data Supplement 2 to the "Guide to the expression of uncertainty in measurement" Extension to any number of output quantities JCGM 102:2011*. BIPM Joint Committee for Guides in Metrology, Paris, Sèvres, 1st edition, 2011. (available on-line here). 37

[22] R Willink. On the validity of methods of uncertainty evaluation. *Metrologia*, 47:80–89, 2010. 41

[23] B D Hall. Evaluating methods of calculating measurement uncertainty. *Metrologia*, 45:L5–L8, 2008. (text available here). 41

[24] B D Hall. Assessing the performance of uncertainty calculations by simulation. In *Proceedings of 74th ARFTG Microwave Measurement Symposium*, (Bloomfield, Co. USA), 2009. (DOI: 10.1109/ARFTG74.2009.5439108; text available here). 41

[25] B D Hall. Monte Carlo uncertainty calculations with small-sample estimates of complex quantities. *Metrologia*, 43:220–226, 2006. 41

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