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Notes on complex measurement uncertainty – part 1

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Summary

This document summarizes a number of topics on the subject of measurement uncertainty in complex quantities with particular emphasis on RF and microwave measurements. The intent is to provide a concise reference.

The full report is in two parts. Part 1 (this document) applies to measurements where certain simplifying assumptions can be made about the associated measurement errors. Under these conditions, the region in the complex plane associated with the uncertainty of a complex measurement result is a circle.

Specifically, it is assumed that

- the measurand and all influence quantities are complex-valued;
- the real and imaginary components of all influences are independent;
- estimates of the real and imaginary components of any quantity are subject to errors of equal variance and the same type of distribution.

Part 2 of the report will deal with a more general treatment of the complex measurement uncertainty problem.

This document covers

- uncertainty reporting
- statistical evaluation of uncertainty (type-A)
- evaluation of uncertainty from other information (type-B)
- propagation of uncertainty

Each section begins with a brief summary of a particular topic, followed by subsections with related material.

A section at the end of the report gives examples of application to power, attenuation and vector network analyser measurements.

This report has also been published as ANAMET Report 053.

Contents

1 Introduction

The evaluation and expression of measurement uncertainty in complex quantities is not fully covered in the *Guide to the Expression of Uncertainty in Measurement* (GUM) [1]. So, part of the international metrology community interested in radio and microwave frequency measurements have developed methods of working with uncertainty in complex quantities, while trying to respect the approach taken in the GUM. This report assembles information that is scattered across journal articles, conference presentations and best-practice guides, for easier reference.

The full report is in two parts. Part 1 (this document) deals with measurements that comply with a fairly restrictive set of assumptions (described in §2.1). These conditions may be satisfied in every-day measurements, so the methods are of practical interest. Section 8 gives examples of application to vector network analyser (VNA) measurements, power measurements and attenuation measurements.

Part 2 deals with the more general bivariate treatment of the complex measurement uncertainty problem. This requires a more elaborate matrix-based approach that introduces a number of multivariate extensions to GUM concepts. In contrast, the methods presented here in Part 1 appear quite similar to the GUM and therefore represent a useful intermediate step towards the more general procedures.

1.1 Using this document

This document is intended primarily for reference. It is hoped that, between the section headings and the index provided at the end of the document, it will be relatively easy to locate relevant information.

The five main sections cover distinct aspects of uncertainty evaluation and reporting and, although there is some cross-referencing between these sections, they can be read in any order.

The additional comments in each section are short notes related to the main section topic but often unrelated to each other. They too can be read in any order.

Here is a short summary of main sections.

Uncertainty statements for complex quantities describes how to determine the size of a circular uncertainty region. Additional comments relate to

- the GUM concept of an expanded uncertainty
- the meaning of the term coverage probability
- the different coverage factors for the one and two-dimensional cases
- the reason for a circle of uncertainty
- uncertainty statements in polar coordinates

Evaluating type-A uncertainty deals with the statistical evaluation of uncertainty. Additional comments relate to

- the GUM concept of type-A uncertainty

- the relationship between a single standard uncertainty, characterising an uncertainty circle, and individual standard uncertainties for the real and imaginary components
- correlation between estimates of the real and imaginary components
- the notion of a covariance matrix to represent complex uncertainty
- the relationship of an uncertainty circle to the more general notion of an uncertainty ellipse

Evaluating type-B uncertainty deals with the evaluation of uncertainty without statistical methods. In particular, how to handle the uncertainty of a complex quantity when phase is unknown. Additional comments relate to

- the GUM concept of type-B uncertainty
- the notion of degrees-of-freedom for a type-B uncertainty
- the problem of associating non-Gaussian distributions with measurement errors

Propagation of uncertainty describes a method for uncertainty propagation with complex quantities and gives simplified rules that apply to arithmetic operations. Additional comments relate to

- the *Law of Propagation of Uncertainty* (LPU) described in the GUM
- the application of simple rules to expressions with mixed arithmetic operations
- the relationship between components of uncertainty in a complex quantity and the GUM notion of components of combined uncertainty in real-valued quantities
- multiple measurands and correlation
- full uncertainty propagation for complex quantities

Degrees of freedom presents an expression for evaluating a number of effective degrees-of-freedom for complex quantities. Additional comments relate to

- the Welch-Satterthwaite formula described in the GUM
- the importance of degrees of freedom
- an approximate expression for the effective degrees of freedom of a type-B uncertainty

1.2 Notation

The following notation is used in this report.

Real-valued quantities and quantity estimates are written in plain italic font, like X or x. Complex-valued quantities and quantity estimates are written in bold italic font, like $\boldsymbol{X}$ or $\boldsymbol{x}$. Greek characters are not italicised, but are also shown in plain style when representing real values and in bold when representing a complex values, e.g., Γ and $\boldsymbol{\Gamma}$.

Matrices are typeset in bold Roman (upright) font, like $\mathbf{v}$.

The imaginary unit j, where $\mathrm{j}^2 = -1$, is used, e.g., $\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}\, x_{\mathrm{im}}$ (note too, the use of subscripts identifying the real and imaginary components).

When describing general mathematical methods, uppercase characters are used to represent physical quantities and lowercase characters denote quantity estimates. However, in many measurement domains this notational convention clashes with standard notations. So, when describing measurement examples the distinction between quantities and estimates is determined by the context.

2 Fundamental assumptions

2.1 Main points

The measurement uncertainty of a complex quantity can be represented as a region in the complex plane around the best estimate of the quantity of interest. In Part 1, the shape of that region is a circle. This is a consequence of adopting a simple model for measurement errors.

Specifically, it is assumed that

- the measurand and all influence quantities are complex-valued;
- the real and imaginary components of all influences are independent;
- estimates of the real and imaginary components of any quantity are subject to errors of equal variance and the same type of distribution.

When these assumptions hold, estimates of real and imaginary components of the measurand are independent and the associated uncertainties are equal.

As in the GUM, it is assumed throughout this report that measurement results are subject to Gaussian errors. In Part 1, it is assumed that errors in the real and imaginary components of a result are independent and have equal variance.

2.2 Additional comments

2.2.1 About errors and uncertainty

In this report, the quantity intended to be measured (the measurand) is considered to have a fixed (unknown) value. A measurement provides an estimate of the measurand that is subject to some (unknown) measurement error (generally the net effect of many different sources of error).

Measurement uncertainty characterises the magnitude of typical measurement errors. In the case of complex quantities, the measurement uncertainty can be represented as a region of the complex plane in which the measurand is likely to be located.

3 Uncertainty statements for complex quantities

3.1 Main points

Figure 1 shows a circular uncertainty region for a quantity $\boldsymbol{X}$ estimated by $\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}x_{\mathrm{im}}$. The circle is centered on $\boldsymbol{x}$ and its area is determined by the required coverage probability p, or level of confidence. The circle radius

$$\boxed{U_{\mathrm{r}} = k_{2,p}u(\boldsymbol{x}) \ ,} \tag{1}$$

where $u(\boldsymbol{x})$ is a standard uncertainty associated with the estimate $\boldsymbol{x}$, and $k_{2,p}$ is a two-dimensional coverage factor that scales the area to provide approximately $100p$ % coverage probability.

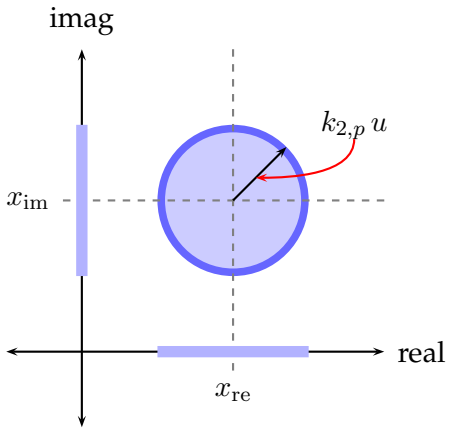

Figure 1: The uncertainty regions in Part 1 of this report are circles, with an area determined by the required the coverage probability.

The coverage factor depends on the required coverage probability and on the number of degrees of freedom ν (degrees-of-freedom are discussed in §4.1 and §7.1),

$$\boxed{k_{2,p}^2 = \frac{2\nu}{\nu - 1}F_{2,\nu-1}(p) \ ,} \tag{2}$$

where $F_{2,\nu-1}(p)$ is the upper $100p^{\mathrm{th}}$ percentile of the F-distribution [2, Ch 5].[1]

If ν is infinite, equation (2) simplifies to

$$\boxed{k_{2,p}^2 = \chi_{2,p}^2 \ ,} \tag{3}$$

where $\chi_{2,p}^2$ is the $100p\%$ point of the chi-square distribution with 2 degrees of freedom.[2]

[1] This is the coverage factor for the conventional elliptical uncertainty region, which will be described in part 2. The actual coverage obtained when this factor is used to scale a circular uncertainty region will tend to be greater than nominal (conservative).

[2] The relation $\chi_{2,p}^2 = -2\log_{\mathrm{e}}(1-p)$ may be useful for calculations.

3.2 Additional comments

3.2.1 Uncertainty statements for real quantities

The measurement uncertainty of a real-valued quantity is expressed as an interval in the GUM [1]. A measurand is considered to lie somewhere between the upper and lower bounds of an uncertainty interval, with a certain coverage probability. The half-width of the interval is called the *expanded uncertainty*

$$U = k_p u \ ,$$

where k_p is the one-dimensional *coverage factor*, p is the desired *coverage probability* and u is the *standard uncertainty* associated with an estimate of the quantity of interest. If x is the best estimate of the quantity of interest, the uncertainty interval is

$$[x - U, x + U] \ .$$

3.2.2 What does coverage probability mean?

Coverage probability, or level of confidence, is a performance measure of the method used to calculate uncertainty. Because measurement errors are unpredictable, there can be no guarantee that an uncertainty interval, or region, covers the measurand for a particular result. Nevertheless, the success-rate of a method of uncertainty calculation can be specified for a large number of independent measurements. This is the coverage probability. For example, an uncertainty calculation with a nominal 95% coverage probability should generate uncertainty statements that cover the measurand on about 95 out of 100 occasions.

3.2.3 Is there a difference between $k_{2,p}$ and k_p?

The two-dimensional coverage factor $k_{2,p}$ is not the same as k_p [1, §6.3]. Table 1 compares the two factors at 95% coverage probability for different numbers of degrees of freedom ν.

Note, the Excel worksheet function `FINV` can be used to calculate the two-dimensional coverage factor. The table entry for $k_{2,0.95}$ with $\nu = 6$ can be found by evaluating

$$\sqrt{\frac{12}{5}\texttt{FINV}(0.05, 2, 5)} = 3.7 \ .$$

The Excel worksheet function `TINV` can be used to calculate the one-dimensional coverage factor. The table entry for $k_{0.95}$ with $\nu = 6$ can be found by evaluating[3]

$$\texttt{TINV}(0.05, 6) = 2.5 \ .$$

[3] Note, the third argument to `FINV` is $\nu - 1 = 5$ in this case.

Table 1: A comparison of 1-D and 2-D coverage factors for $p = 0.95$ and different degrees-of-freedom ν. The 1-D coverage factor k_p is used for real-valued problems, where an uncertainty interval is required, and the 2-D factor $k_{2,p}$ is used for complex-valued problems, where an uncertainty region is required.

ν	$k_{2,0.95}$	$k_{0.95}$	ν	$k_{2,0.95}$	$k_{0.95}$
2	28.3	4.3	7	3.5	2.4
3	7.6	3.2	8	3.3	2.3
4	5.1	2.8	9	3.2	2.3
5	4.2	2.6	10	3.1	2.2
6	3.7	2.5	50	2.6	2.0
			∞	2.45	1.96

3.2.4 Why are different coverage factors needed?

A complex quantity has two components, so information about the uncertainty of each component-estimate must be included in a complex uncertainty statement.

Suppose that $\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}x_{\mathrm{im}}$ is a measurement result. The intervals associated with real and imaginary component uncertainties can be visualised as orthogonal bands in the complex plane, as shown in Figure 2.

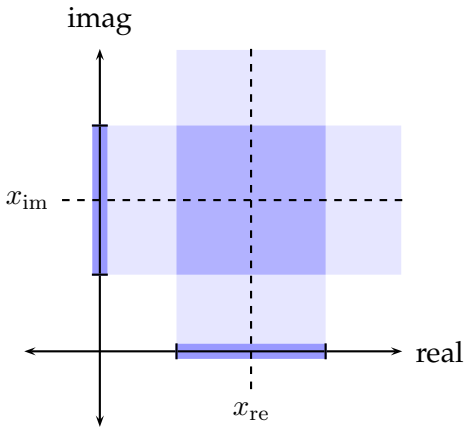

Figure 2: A superposition of uncertainty intervals for real and imaginary components

The band intersection is of interest in the complex problem, because it represents a set of complex values close to the best estimate. If the measurement were to be repeated independently many times (generating different observations and hence different bands), the intersection would cover the complex measurand less often than an individual band would cover the associated real or imaginary component. In other words, the coverage probability of the two-dimensional band intersection is lower than the coverage probability of one band (which is really a one-dimensional interval for one of the components). Therefore, to achieve the desired coverage probability, the two-dimensional coverage factor has to be larger than the one-dimensional factor.

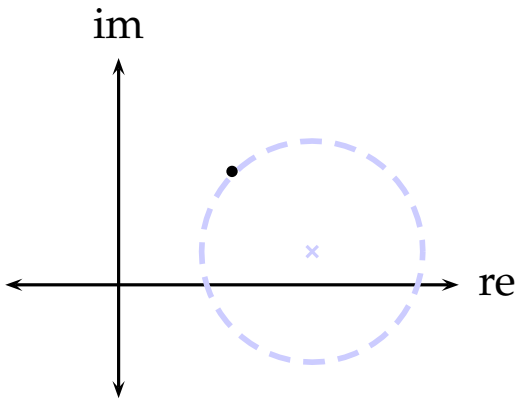

Figure 3: The dashed line is a contour of constant error density (the circle radius is δ) associated with measurements of $\boldsymbol{X}$. The measurand is located at the cross and the dot is one observation of $\boldsymbol{X}$.

3.2.5 Why is the uncertainty region a circle?

In answering this question, it is important to keep in mind the quite different nature of the measurand $\boldsymbol{X}$ (a fixed, but unknown quantity), the measurement error $\boldsymbol{e}$ (an unknown random quantity) and the measurement result, or estimate, $\boldsymbol{x}$ (a number, that varies from one measurement to the next). The estimate is related to the measurand by

$$\boldsymbol{x} = \boldsymbol{e} - \boldsymbol{X} \ .$$

For this discussion, we presume to know the type and scale of the error distribution, although $\boldsymbol{X}$ is unknown. In particular, we assume that contours of constant error density are circles, centered on $\boldsymbol{X}$. Our task is to consider all possible values of $\boldsymbol{X}$ that are compatible with an observation $\boldsymbol{x}$.

Let there be a 95% probability that

$$|\boldsymbol{X} - \boldsymbol{x}'| <= \delta \ ,$$

where $\boldsymbol{x}'$ is some measurement of $\boldsymbol{X}$ and δ is a constant. The locus of points $\boldsymbol{x}'$ that satisfy this inequality is a disk of radius δ centered on $\boldsymbol{X}$.

Now, a particular choice of $\boldsymbol{X}$, for which $\boldsymbol{x}$ lies on the circumference, is shown in Figure 3. In this scenario, we would not reject $\boldsymbol{x}$ as being an observation of $\boldsymbol{X}$ with a level of confidence of 95%.[4]

To construct an uncertainty region containing all reasonable values of $\boldsymbol{X}$, we should consider the $\boldsymbol{X}$'s at the centres of all disks of radius δ that cover $\boldsymbol{x}$. This generates a circle around $\boldsymbol{x}$.

In general, what is the shape of the uncertainty region? A more general assumption than those used in Part 1 of this report is that a bivariate Gaussian distribution of errors affects the measurement results. This extends the GUM's

[4]However, if $\boldsymbol{X}$ was further away, our confidence would be reduced.

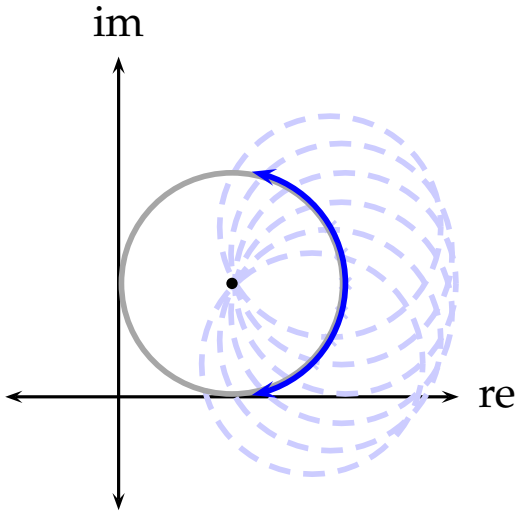

Figure 4: The dashed circles are error contours, as in Fig. 3. Each circle is centered on a different possible value of the measurand $\boldsymbol{X}$. The locus of all possible $\boldsymbol{X}$ values is a disk centered on the observation x, shown as a dot. The solid blue line is intended to represent a partially constructed perimeter, the grey circle is the complete uncertainty region.

assumption, that the distribution of errors associated with a real-valued measurement is approximately Gaussian, and can be justified by appeal to the multivariate Central Limit Theorem [2, Ch. 4].

The conventional shape of an uncertainty region in this case is an ellipse, arbitrarily oriented with respect to the real-imaginary coordinate system (a generalisation of §3.2.5 can explain this shape too).

The coverage factor $k_{2,p}$ scales the size of the ellipse to obtain an uncertainty region with the desired coverage probability.

3.2.6 Statements of uncertainty in polar coordinates

It is sometimes possible to transform a region of uncertainty expressed in rectangular coordinates into one expressed in polar coordinates. Similarly, when uncertainty information is required in rectangular coordinates, it may be possible to transform a statement given in polar coordinates. However, the transformation is non-linear and will not always be reliable: when the uncertainty region lies close to the origin, the uncertainty region is distorted and the coverage probability is difficult to predict.

Rectangular to polar: Suppose that the standard uncertainty u is associated with a circular uncertainty region around a point $\boldsymbol{x}$. Provided $u/|\boldsymbol{x}| \ll 1$, the standard uncertainty in the radial coordinate is

$$u(|\boldsymbol{x}|) = u$$

and the standard uncertainty in the phase coordinate as

$$u(\phi) \approx \frac{u}{|\boldsymbol{x}|} \,,$$

where $\phi \approx \arg(\boldsymbol{x})$.

The geometry of the problem is shown in Figure 5. The radius of the circle u is the standard uncertainty associated with x_{re} and x_{im}. The uncertainty along the radial direction is also u. However, the phase uncertainty $u(\phi)$ is open to different interpretations.[5] Here, we have followed [5].

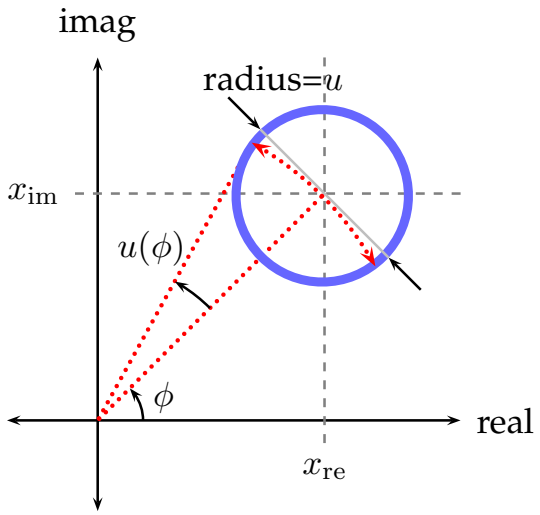

Figure 5: Geometry of uncertainty region for coordinate transformation with a circle radius u equal to the standard uncertainty.

Polar to rectangular: If an approximately circular uncertainty region is defined in polar coordinates,[6] the standard uncertainty associated with estimates of the real and imaginary components of $\boldsymbol{x}$ is equal to the circle radius divided by the two-dimensional coverage factor for the region.

[5]Other suggestions include $u(\phi) = \tan^{-1} u/|\boldsymbol{x}|$ [3] and $u(\phi) = \sin^{-1} u/|\boldsymbol{x}|$ [4]. Simulation studies have shown that there is little practical difference between these choices. Instead, it is more important to respect the requirement that $u/|\boldsymbol{x}| \ll 1$. All the approximations for $u(\phi)$ break down at similar values of $u/|\boldsymbol{X}|$.

[6]I.e., if $|\boldsymbol{x}|\, u(\phi) \approx u(|\boldsymbol{x}|)$.

4 Evaluating type A uncertainty

4.1 Main points

The best estimate of a complex quantity $\boldsymbol{X}$, based on a sample of N observations $\boldsymbol{x}_1 \cdots \boldsymbol{x}_N$, is usually the arithmetic mean (written here without the 'bar' notation)

$$\boldsymbol{x} = x_{\mathrm{re}} + \mathrm{j}\, x_{\mathrm{im}} \ ,$$

where

$$x_{\mathrm{re}} = \frac{1}{N} \sum_{i=1}^{N} x_{\mathrm{re}\cdot i} \ ,$$

$$x_{\mathrm{im}} = \frac{1}{N} \sum_{i=1}^{N} x_{\mathrm{im}\cdot i}$$

and the individual observations are

$$\boldsymbol{x}_i = x_{\mathrm{re}\cdot i} + \mathrm{j}\, x_{\mathrm{im}\cdot i} \ .$$

The standard uncertainty associated with *both* the real and imaginary components of $\boldsymbol{x}$ is

$$\boxed{u(\boldsymbol{x}) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} \frac{|\boldsymbol{x} - \boldsymbol{x}_i|^2}{2}} \ ,} \tag{4}$$

or equivalently

$$\boxed{u(\boldsymbol{x}) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} \frac{(x_{\mathrm{re}} - x_{\mathrm{re}\cdot i})^2 + (x_{\mathrm{im}} - x_{\mathrm{im}\cdot i})^2}{2}} \ .} \tag{5}$$

The associated degrees-of-freedom

$$\boxed{\nu = N - 1 \ .} \tag{6}$$

The three numbers, $\boldsymbol{x}$, $u(\boldsymbol{x})$ and ν, characterise the measurement result, which provides an estimate of $\boldsymbol{X}$.

4.2 Additional comments

4.2.1 Type A uncertainty for real quantities

As described in the GUM [1, §4.2.1], the best estimate of a quantity of interest X, based on a sample of N observations $x_1 \cdots x_N$, is usually the arithmetic mean (written here without the 'bar' notation)

$$x = \frac{1}{N} \sum_{i=1}^{N} x_i \ .$$

The associated standard uncertainty is the standard deviation of the sample mean,

$$u(x) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} (x - x_i)^2}$$

and the number of degrees of freedom is

$$\nu = N - 1 \ .$$

In the GUM, the numbers x, $u(x)$ and ν characterise a measurement result.

4.2.2 But the uncertainty in the real and imaginary components may not be equal!

Conventional data processing of a sample of N observations $\boldsymbol{x}_1 \cdots \boldsymbol{x}_N$ would evaluate distinct values of the standard uncertainty for the real and imaginary components, i.e.

$$u(x_{\mathrm{re}}) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} (x_{\mathrm{re}} - x_{\mathrm{re}\cdot i})^2}$$

and

$$u(x_{\mathrm{im}}) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} (x_{\mathrm{im}} - x_{\mathrm{im}\cdot i})^2} \ .$$

These standard uncertainties will not, in general be equal. The standard uncertainty obtained in §4.1 is the root-mean-square of these values

$$u(\boldsymbol{x}) = \sqrt{\frac{u(x_{\mathrm{re}})^2 + u(x_{\mathrm{im}})^2}{2}} \ .$$

In effect, $u(\boldsymbol{x})$ is a summary value for $u(x_{\mathrm{re}})$ and $u(x_{\mathrm{im}})$. That is, $u(\boldsymbol{x})$ characterises a circular uncertainty region that is an approximation to the more general elliptical uncertainty region (see also §4.2.6).

4.2.3 What about correlation?

The assumptions made in Part 1 of this report effectively postulate zero correlation between the real and imaginary component estimates.

However, in the GUM, the sample covariance between x_{re} and x_{im} can be evaluated as

$$u(x_{\mathrm{re}}, x_{\mathrm{im}}) = \frac{1}{N(N-1)} \sum_{i=1}^{N} (x_{\mathrm{re}} - x_{\mathrm{re}\cdot i})(x_{\mathrm{im}} - x_{\mathrm{im}\cdot i})$$

and the sample correlation coefficient between the components of $\boldsymbol{x}$ is

$$r(x_{\mathrm{re}}, x_{\mathrm{im}}) = \frac{u(x_{\mathrm{re}}, x_{\mathrm{im}})}{u(x_{\mathrm{re}})\, u(x_{\mathrm{im}})} \ .$$

In Part 2 of this report, information about correlation is incorporated in type-A uncertainty statements. However, in Part 1 it is ignored.

4.2.4 What is the covariance matrix?

In the more general treatment of uncertainty described in Part 2, a covariance matrix conveniently groups together information about standard uncertainties and correlation between the real and imaginary component estimates. The matrix takes the form

$$\mathbf{v}(\boldsymbol{x}) = \begin{bmatrix} u(x_{\mathrm{re}})^2 & u(x_{\mathrm{re}}, x_{\mathrm{im}}) \\ u(x_{\mathrm{re}}, x_{\mathrm{im}}) & u(x_{\mathrm{im}})^2 \end{bmatrix}$$

and can always be factored into

$$\mathbf{u}\,\mathbf{r}\,\mathbf{u}\ ,$$

where

$$\mathbf{u} = \begin{bmatrix} u(x_{\mathrm{re}}) & 0 \\ 0 & u(x_{\mathrm{im}}) \end{bmatrix}$$

is a matrix of standard uncertainties and

$$\mathbf{r} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}\ .$$

In Part 1, the uncertainties in the real and imaginary components are equal, and there is no correlation, so the covariance matrix always has the simple diagonal form

$$\mathbf{v} = \begin{bmatrix} u(\boldsymbol{x})^2 & 0 \\ 0 & u(\boldsymbol{x})^2 \end{bmatrix} \quad = \quad u(\boldsymbol{x})^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ .$$

So, there is little reason to use a matrix representation in this part of the report.

4.2.5 Avoid polar coordinates when evaluating type-A uncertainty

When using sample statistics to evaluate the uncertainty of a complex quantity, *never* work in polar coordinates! Potentially serious problems can occur.

1. The statistics of the transformed sample data will, in general, be different from those of the data before transformation, because of the non-linearity of the transformation between rectangular and polar coordinates.
2. The periodicity of phase values is difficult to handle correctly.

These concerns are discussed in detail in [6] and [7].

4.2.6 How is an uncertainty circle related to an uncertainty ellipse?

If a type-A uncertainty is evaluated according to §4.2.2 and §4.2.3, different standard uncertainties for the real and imaginary components, and a non-zero sample correlation coefficient, will generally be obtained. The conventional shape of an uncertainty region in that case is an ellipse.

However, if the method of §4.1 is applied to the same data, a single standard uncertainty is obtained and a circular region is used to represent the uncertainty.

The area of the uncertainty circle associated with the method of §4.1 is the mean of the inscribed and circumscribed circle areas for the corresponding uncertainty ellipse (Figure 6). So, a circular uncertainty region should be thought of as an approximation to an elliptical region.

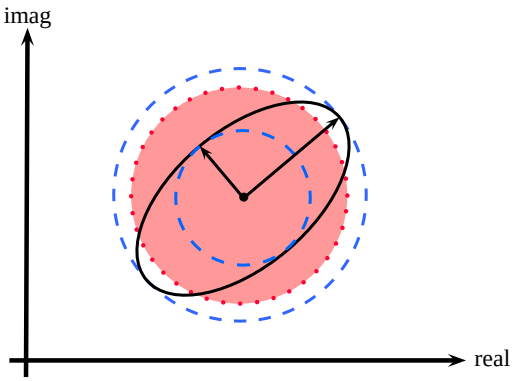

Figure 6: The area of the coloured circle is the mean of the inscribed and circumscribed circle areas (dashed circles) of the uncertainty ellipse.

Although it conveys less information than an ellipse, a circle does perform fairly well as the uncertainty region. Simulations have shown that the coverage probability of the circular uncertainty regions tends to be conservative (coverage probability tends to be above nominal). Note, too, that the area of the uncertainty circle does not go to zero if a very small standard uncertainty is obtained for one of the components.

5 Evaluating type B uncertainty

5.1 Main points

In RF and microwave measurements, a type-B uncertainty arises when nothing is known about the phase of an influence quantity. Three cases are considered below, in which different information is available about the magnitude.

In addition, a type-B uncertainty is associated with a complex product of terms when the phase of one, or both, of the factors is unknown. Such a product needs to be treated as an independent influence quantity, because the methods of uncertainty propagation described in §6.1 do not work in this case. A simple relation between the uncertainty of the product and the uncertainties of the factors is given below.

5.1.1 Known magnitude

When the magnitude $|\mathbf{\Gamma}| = a$ is known, $\mathbf{\Gamma}$ could be attributed to any point on a circle around the origin.

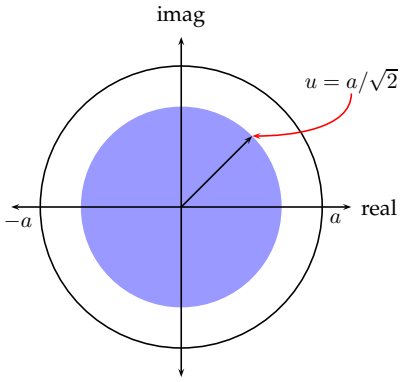

Figure 7: The uncertainty region is a ring when the magnitude is known. The radius $a = |\mathbf{\Gamma}|$ and the standard uncertainty $u(\mathbf{\Gamma}) = a/\sqrt{2}$.

This corresponds to a *uniform ring* of uncertainty. The standard uncertainty in this case is [3]

$$\boxed{u(\mathbf{\Gamma}) = \frac{a}{\sqrt{2}}\,.}$$

5.1.2 Bounded magnitude

When an upper limit for the magnitude is known $|\mathbf{\Gamma}| \leq a$, $\mathbf{\Gamma}$ could be attributed to any point in a circular region around the origin (Figure 8). This corresponds to a *uniform disk* of uncertainty. The standard uncertainty in this case is [3]

$$\boxed{u(\mathbf{\Gamma}) = \frac{a}{2}\,.}$$

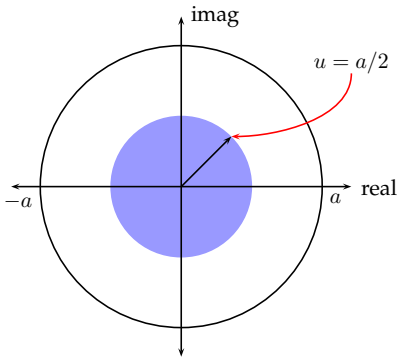

Figure 8: The uncertainty region is a disk when the magnitude is bounded above. The radius $a = |\mathbf{\Gamma}|$ and the standard uncertainty $u(\mathbf{\Gamma}) = a/2$.

5.1.3 Magnitude estimate

When the magnitude is estimated $|\mathbf{\Gamma}| \approx a$, and the estimate has an uncertainty $u(a)$, $\mathbf{\Gamma}$ could be attributed to any point in a radially symmetric distribution around the origin. The standard uncertainty in this case is [8][7]

$$\boxed{u(\mathbf{\Gamma}) = \sqrt{\frac{a^2 + 2u^2}{2}}\,.}$$

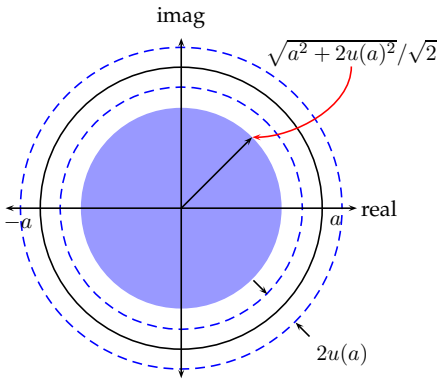

Figure 9: When the radius is estimated $a \approx |\mathbf{\Gamma}|$, with an uncertainty $u(a)$, the standard uncertainty $u(\mathbf{\Gamma}) = \sqrt{a^2 + 2u^2}/\sqrt{2}$. The geometry of the uncertainty region can be crudely represented as a uniform annulus.

The distribution associated with the uncertainty in this case does not have radial bounds. However, most of its density is located close to the circle of radius a. So, the geometry of the uncertainty region can be crudely represented as a *uniform annulus* of width $2u(a)$, which would give the same standard uncertainty (Figure 9).

[7] The UKAS Guide M3003 uses a different expression $u = \sqrt{a^2 + u(a)^2}$ (see [9, E9.6]). Simulations suggest that there is little practical difference in the coverage probabilities obtained using either formula.

5.1.4 Product of estimates

When a product of two complex quantities

$$\mathbf{\Gamma} = \mathbf{\Gamma}_1 \mathbf{\Gamma}_2$$

influences a measurement and the phase of one or both factors is unknown, the phase of $\mathbf{\Gamma} \approx 0$ will be unknown. In this case, the product $\mathbf{\Gamma}$ could be attributed to any point in radially symmetric distribution around the origin. The standard uncertainty is

$$\boxed{u(\mathbf{\Gamma}) = \sqrt{2} u(\mathbf{\Gamma}_1) u(\mathbf{\Gamma}_2) \ .}$$

Unfortunately, it only takes one factor with an unknown phase to generate a product with unknown phase. Effectively, all phase information about the known factor $\mathbf{\Gamma}_1$ is lost in the product. In such cases, consideration should be given to what is known about the magnitude of the factor and an appropriate type-B uncertainty should be chosen on that basis.

5.2 Additional comments

5.2.1 Type B uncertainty for real quantities

The GUM describes a number of type-B uncertainty distributions: uniform, triangular and arcsine (or U-shaped). It is interesting to note that the marginal distribution of the uniform ring along the real or imaginary axes is the arcsine distribution.

5.2.2 Can degrees-of-freedom be associated with a type-B uncertainty?

The GUM provides an expression relating a number of degrees-of-freedom to what is judged to be the relative uncertainty of a type-B evaluation of uncertainty [1, G.4.2]. A similar expression is applicable to the symmetrical type-B distributions considered here, see §7.2.3.

5.2.3 These distributions are very different from Gaussian

One of the key assumptions underpinning the evaluation of uncertainty is that the error distribution associated with measurement results is approximately Gaussian. The type-B distributions described in this section are not at all Gaussian-like, so it is important be aware just how much they influence the uncertainty associated with a result.

It is to be expected from the Central Limit Theorem that the net effect of many smaller contributing influences will be an approximately Gaussian error in the measurement result.

A recent study has investigated the performance of uncertainty calculations in which this type of type-B influences are involved [8]. In general terms, the study found that uncertainty calculation performance remained satisfactory provided the contribution from these type-B influences does not dominate the uncertainty budget.

6 Propagation of uncertainty

6.1 Main points

The propagation of uncertainty from influence quantities to a measurement result requires analysis of an equation describing the measurement procedure (including data-processing). This equation can be expressed as

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N) \ ,$$

where all quantities are complex-valued and the function $\boldsymbol{f}$ is analytic.[8] Estimates of the influence quantities are

$$\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$$

so an estimate of the measurand is

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N) \ .$$

A standard uncertainty is associated with each estimate, denoted here as $u(\boldsymbol{x}_i)$.

A component of uncertainty in $\boldsymbol{y}$ due to uncertainty in $\boldsymbol{x}_i$ is defined as[9]

$$\boxed{u_i(\boldsymbol{y}) = \left| \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_i} \right| \ u(\boldsymbol{x}_i) \ .} \tag{7}$$

The combined standard uncertainty associated with $\boldsymbol{y}$ is[10]

$$\boxed{u(\boldsymbol{y}) = \left[\sum_{i=1}^{N} u_i(\boldsymbol{y})^2 \right]^{1/2} .} \tag{8}$$

The combined standard uncertainty is associated with both the real and imaginary components of $\boldsymbol{y}$ [10].

6.1.1 Special cases – independent estimates

When a measurement equation is composed of simple arithmetic operations, and all influence quantities are mutually independent, there are simple rules for evaluating the combined standard uncertainty.

Addition and subtraction: For an equation of the form

$$\boldsymbol{y} = \boldsymbol{x}_1 \oplus \boldsymbol{x}_2 \oplus \cdots \ ,$$

where each $\oplus$ may represent either '$+$' or '$-$', the combined standard uncertainty $u(\boldsymbol{y})$ is the root-sum-square of the components

$$u(\boldsymbol{y}) = \left[u(\boldsymbol{x}_1)^2 + u(\boldsymbol{x}_2)^2 + \cdots \right]^{1/2} \ .$$

[8] The partial derivatives of $\boldsymbol{f}$ must be well-defined at the estimates $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$.

[9] The GUM defines a 'component of combined standard uncertainty' [1, J]. The concepts are closely related and we hope that our use of notation here, which places a complex quantity between the parentheses, will be sufficient to distinguish between them.

[10] This equation requires that all influence quantity estimates are independent.

Multiplication and division: For an equation of the form

$$\boldsymbol{y} = \boldsymbol{x}_1 \otimes \boldsymbol{x}_2 \otimes \cdots \ ,$$

where each $\otimes$ may be either '$\times$' or '$\div$', the combined standard uncertainty $u(\boldsymbol{y})$ can be found from

$$\frac{u(\boldsymbol{y})}{|\boldsymbol{y}|} = \left[\left(\frac{u(\boldsymbol{x}_1)}{|\boldsymbol{x}_1|} \right)^2 + \left(\frac{u(\boldsymbol{x}_2)}{|\boldsymbol{x}_2|} \right)^2 + \cdots \right]^{1/2} \ .$$

Powers: When a measurement equation is of the form

$$y = \boldsymbol{x}^n \ ,$$

where n is a real number, the combined standard uncertainty can be found from

$$\frac{u(\boldsymbol{y})}{|\boldsymbol{y}|} = n \frac{u(\boldsymbol{x})}{|\boldsymbol{x}|} \ .$$

6.1.2 Special cases – systematic errors

A systematic error is sometimes represented by using several fully correlated terms in a measurement equation. The rules for combining uncertainties are different in this case.

In the following it is assumed that the estimates $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots$ are perfectly correlated, i.e., that the correlation coefficient $r = 1$ between pairs of real components and between pairs of imaginary components (§4.2.3).

Addition: For an equation of the form

$$\boldsymbol{y} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots \ ,$$

the combined standard uncertainty $u(\boldsymbol{y})$ is the sum of the components

$$u(\boldsymbol{y}) = u(\boldsymbol{x}_1) + u(\boldsymbol{x}_2) + \cdots \ .$$

Subtraction: For an equation of the form

$$\boldsymbol{y} = \boldsymbol{x}_1 - \boldsymbol{x}_2 - \cdots \ ,$$

the combined standard uncertainty $u(\boldsymbol{y})$ is the difference of the components

$$u(\boldsymbol{y}) = u(\boldsymbol{x}_1) - u(\boldsymbol{x}_2) - \cdots \ .$$

Multiplication: For an equation of the form

$$\boldsymbol{y} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \cdots \ ,$$

the combined standard uncertainty $u(\boldsymbol{y})$ can be found from

$$\frac{u(\boldsymbol{y})}{|\boldsymbol{y}|} = \frac{u(\boldsymbol{x}_1)}{|\boldsymbol{x}_1|} + \frac{u(\boldsymbol{x}_2)}{|\boldsymbol{x}_2|} + \cdots \ .$$

Division: For an equation of the form

$$\boldsymbol{y} = \boldsymbol{x}_1 \div \boldsymbol{x}_2 \div \cdots ,$$

the combined standard uncertainty $u(\boldsymbol{y})$ can be found from

$$\frac{u(\boldsymbol{y})}{|\boldsymbol{y}|} = \frac{u(\boldsymbol{x}_1)}{|\boldsymbol{x}_1|} - \frac{u(\boldsymbol{x}_2)}{|\boldsymbol{x}_2|} - \cdots .$$

6.2 Additional comments

6.2.1 Expressions with mixed arithmetic operations

When a measurement equation contains a mixture of different arithmetic operations, it may be possible to decompose the problem into a number of simpler expressions that can be handled by the rules in §6.1.1.

For example,

$$\boldsymbol{Y} = \boldsymbol{X}_1(\boldsymbol{X}_2 - \boldsymbol{X}_3) + \boldsymbol{X}_4$$

can be decomposed into three steps:

$$\begin{aligned} \boldsymbol{Y}_1 &= \boldsymbol{X}_2 - \boldsymbol{X}_3 \\ \boldsymbol{Y}_2 &= \boldsymbol{X}_1 \boldsymbol{Y}_1 \\ \boldsymbol{Y} &= \boldsymbol{Y}_2 + \boldsymbol{X}_4 \end{aligned}$$

- the rule for subtraction applied to the first step gives

$$u(\boldsymbol{y}_1)^2 = u(\boldsymbol{x}_2)^2 + u(\boldsymbol{x}_3)^2$$

- the product rule would be used with the second step giving

$$u(\boldsymbol{y}_2)^2/|\boldsymbol{y}_2|^2 = u(\boldsymbol{x}_1)^2/|\boldsymbol{x}_1|^2 + u(\boldsymbol{y}_1)^2/|\boldsymbol{y}_1|^2$$

- the addition rule would be used with the final step to give

$$u(\boldsymbol{y})^2 = u(\boldsymbol{y}_2)^2 + u(\boldsymbol{x}_4)^2$$

Finally, the combined standard uncertainty is

$$u(\boldsymbol{y}) = \sqrt{|\boldsymbol{x}_2 - \boldsymbol{x}_3|^2 \, u(\boldsymbol{x}_1)^2 + |\boldsymbol{x}_1|^2 \left[u(\boldsymbol{x}_2)^2 + u(\boldsymbol{x}_3)^2\right] + u(\boldsymbol{x}_4)^2} \, .$$

Note: This process of decomposition is correct *only* when the intermediate results are independent of each other. For example, if

$$\begin{aligned} \boldsymbol{Y}_1 &= \boldsymbol{X}_1 + \boldsymbol{X}_2 \\ \boldsymbol{Y}_2 &= \boldsymbol{X}_2 + \boldsymbol{X}_3 \\ \boldsymbol{Y}_3 &= \boldsymbol{Y}_1 + \boldsymbol{Y}_2 \end{aligned}$$

then the estimates $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are correlated, because both use $\boldsymbol{x}_2$. So, the standard uncertainty of $\boldsymbol{y}_3 = \boldsymbol{y}_1 + \boldsymbol{y}_2$ is

$$u(\boldsymbol{y}_3) = \sqrt{u(\boldsymbol{x}_1)^2 + 4u(\boldsymbol{x}_2)^2 + u(\boldsymbol{x}_3)^2}$$

and *not*

$$u(\boldsymbol{y}_3) = \sqrt{u(\boldsymbol{x}_1)^2 + 2u(\boldsymbol{x}_2)^2 + u(\boldsymbol{x}_3)^2} \, ,$$

which is the result of treating $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ as independent and using the rules for simple arithmetic.

6.2.2 Propagation of uncertainty for real quantities

The method of propagating uncertainty in §6.1 is similar to the *Law of Propagation of Uncertainty* (LPU) described in the GUM [1, §5]. We can describe the LPU as follows.

An equation among physical quantities describes a measurement

$$Y = f(X_1, X_2, \cdots, X_N) \,,$$

where all quantities are real-valued and the function f is analytic. Estimates of the influence quantities are

$$x_1, x_2, \cdots, x_N$$

so an estimate of the measurand is

$$y = f(x_1, x_2, \cdots, x_N) \,.$$

A standard uncertainty is associated with each estimate, denoted here as $u(x_i)$.

A *component of combined standard uncertainty* in y due to uncertainty in x_i is

$$u_i(y) = \left| \frac{\partial y}{\partial x_i} \right| u(x_i) \,.$$

The combined standard uncertainty associated with y is

$$u(y) = \left[\sum_{i=1}^{N} u_i(y)^2 \right]^{1/2} \,,$$

provided that the estimates $x_1, x_2, \cdots, x_N$ are independent (see [1, §5.1.2]).

6.2.3 How is $u_i(\boldsymbol{y})$ related to components of combined standard uncertainty?

The real and imaginary components of the estimate $\boldsymbol{y}$ are affected by errors in the real and imaginary components of each estimate $\boldsymbol{x}_i$. There are two components of combined standard uncertainty associated with the real component of $\boldsymbol{y}$

$$u_{i\cdot\mathrm{re}}(y_\mathrm{re}) \quad \text{and} \quad u_{i\cdot\mathrm{im}}(y_\mathrm{re})$$

and two components of combined standard uncertainty associated with the imaginary component

$$u_{i\cdot\mathrm{re}}(y_\mathrm{im}) \quad \text{and} \quad u_{i\cdot\mathrm{im}}(y_\mathrm{im}) \,.$$

Individually, these components may be thought of as the amount that y_re, or y_re, would change if $x_{i\cdot\mathrm{re}}$, or $x_{i\cdot\mathrm{im}}$, were to change by an amount equal to $u(\boldsymbol{x}_i)$.

It can be shown that

$$u_i(\boldsymbol{y}) = \sqrt{\frac{u_{i\cdot\mathrm{re}}(y_\mathrm{re})^2 + u_{i\cdot\mathrm{re}}(y_\mathrm{im})^2 + u_{i\cdot\mathrm{im}}(y_\mathrm{re})^2 + u_{i\cdot\mathrm{im}}(y_\mathrm{im})^2}{2}} \,.$$

So the component of uncertainty $u_i(\boldsymbol{y})$ that arises from the propagation of complex measurement uncertainty is a summary value for the four components of combined standard uncertainty that would be obtained if the same problem were broken down into real and imaginary components and analysed using the GUM LPU.

6.2.4 Measurement equations containing a mixture of real and complex quantities

There are many situations where the measurement equation for a real-valued measurand contains complex terms (e.g., in the measurement of power or attenuation). In that case, the methods described can be applied to parts of the main measurement equation that describe complex quantities. These intermediate results can then be expressed in terms of real and imaginary components and the uncertainty analysis completed using the GUM LPU. Examples are given in §8.

6.2.5 Multiple measurands and logical correlation

In some situations there is more than one quantity of interest and several equations are applied to a common set of measurements, leading to correlation in the estimates of each measurand. Such correlation is incompatible with the assumptions made in Part 1. So, such results cannot be propagated in subsequent calculations using the methods described here.

To make this issue clearer, suppose quantities $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$ are measured, giving estimates $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. If the two quantities of interest are $\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \boldsymbol{X}_2$ and $\boldsymbol{Y}_2 = \boldsymbol{X}_2 + \boldsymbol{X}_3$, for which we obtain estimates $\boldsymbol{y}_1 = \boldsymbol{x}_1 + \boldsymbol{x}_2$ and $\boldsymbol{y}_2 = \boldsymbol{x}_2 + \boldsymbol{x}_3$, then $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are correlated, because they both depend on $\boldsymbol{x}_2$.

As a practical example, reflectometer calibration estimates three complex error terms (directivity, tracking and match), from three measurements and three nominal values for the calibration standards used. The error estimates obtained are correlated. So, the more detailed methods described in Part 2 are required to correctly evaluate the uncertainty associated with error-corrected measurements.

In general, this type of correlation arises from the form of the measurement equations. It is sometimes called *logical correlation* and can be evaluated as follows. For a pair of estimates

$$\begin{aligned} \boldsymbol{y}_a &= \boldsymbol{f}_a(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_l) \\ \boldsymbol{y}_b &= \boldsymbol{f}_b(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m) \end{aligned}$$

a complex-valued correlation parameter

$$\boldsymbol{q}_{ab} = \frac{\sum_{i=1}^{l} \sum_{j=1}^{m} \frac{\partial \boldsymbol{f}_a}{\partial \boldsymbol{x}_i} u(\boldsymbol{x}_i) \frac{\partial \boldsymbol{f}_b}{\partial \boldsymbol{x}_j} u(\boldsymbol{x}_j)}{u(\boldsymbol{y}_a)\, u(\boldsymbol{y}_b)}$$

is representative of the four correlation coefficients between the real and imaginary components of $\boldsymbol{y}_a$ and $\boldsymbol{y}_b$. The components of $\boldsymbol{q}_{ab} = q_{ab\cdot\mathrm{re}} + \mathrm{j} q_{ab\cdot\mathrm{im}}$ relate to the real-valued correlation coefficients as follows:

$$\begin{aligned} r(y_{a\cdot\mathrm{re}}, y_{b\cdot\mathrm{re}}) &= q_{ab\cdot\mathrm{re}} \\ r(y_{a\cdot\mathrm{re}}, y_{b\cdot\mathrm{im}}) &= -q_{ab\cdot\mathrm{im}} \\ r(y_{a\cdot\mathrm{im}}, y_{b\cdot\mathrm{re}}) &= q_{ab\cdot\mathrm{im}} \\ r(y_{a\cdot\mathrm{im}}, y_{b\cdot\mathrm{im}}) &= q_{ab\cdot\mathrm{re}}\ . \end{aligned}$$

In Part 2, correlation is dealt with in more detail. The main point here is that the assumptions required to apply the methods of Part 1 are not satisfied unless $\boldsymbol{q}_{ab} = 0$.

6.2.6 Full propagation of uncertainty for complex quantities

It is possible to extend the LPU to more general multivariate measurement problems. An early description of this was given by Weise [11], but Riddler and Salter carefully considered the specific needs of RF and microwave measurements and outlined a bivariate formulation for complex quantities in a series of papers and conference presentations [6, 7, 12].

7 Degrees of freedom

7.1 Main points

Following the description in §6.1, an equation among physical quantities describes a measurement procedure

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N) \ ,$$

where all quantities are complex-valued. An estimate of the measurand

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N) \ .$$

is associated with a standard uncertainty $u(\boldsymbol{y})$. Each input estimate $\boldsymbol{x}_i$ is associated with a standard uncertainty $u(\boldsymbol{x}_i)$ and a number of degrees of freedom ν_i, which may be infinite.

A number of effective degrees of freedom ν_{eff} can be associated with the uncertainty of $\boldsymbol{y}$. This is found from

$$\boxed{\frac{u(\boldsymbol{y})^4}{\nu_{\text{eff}}} = \sum_{i=1}^{N} \frac{u_i(\boldsymbol{y})^4}{\nu_i} \ ,} \tag{9}$$

where $u_i(\boldsymbol{y})$ is a component of uncertainty in $\boldsymbol{y}$ due to uncertainty in $\boldsymbol{x}_i$ (see equation 7).

Note that ν_{eff} is associated with measurement of a *complex quantity* and is intended for use in the calculation of a *complex* coverage factor $k_{p,2}$, to construct an uncertainty region for $\boldsymbol{Y}$ (see §3.1).

7.2 Additional comments

7.2.1 The Welch-Satterthwaite formula

The form of equation (9) is similar to the Welch-Satterthwaite formula (WS) in the GUM [1, G.4]. However, the two calculations are different. WS *only* applies to real-valued problems.

Equation (9) is a special case of the more general 'total variance' calculation described in [13]. The calculation is simplified here because of the assumptions that apply in Part 1.

7.2.2 Are degrees-of-freedom important?

Degrees-of-freedom is a measure of the sample size used to estimate a standard uncertainty. When the degrees-of-freedom is small, the standard uncertainty obtained from a sample of data may be much smaller than the standard deviation of the underlying errors.

To ensure satisfactory coverage probability when the degrees-of-freedom is small, the area of the uncertainty region must be increased (see §3.1). A quick look at Table 1 shows that the coverage factor $k_{2,p}$ is larger than k_p for the same sample size. As a consequence, the degrees-of-freedom is *more* important in complex measurements

than it is for measurements of real-valued quantities. It is unfortunate that the notion of degrees-of-freedom seems to have fallen from favour in recent years. It should not be overlooked.

7.2.3 Effective degrees-of-freedom for type-B uncertainties

There is an expression in the GUM to evaluate an effective degrees-of-freedom, which can be used in association with a type-B uncertainty when there is some doubt about the true width of the associated error distribution.

An analogous expression can be derived for the complex problem when dealing with a radially symmetric distribution centered on the origin (the unknown phase problem). In that case [8]

$$\nu_{\text{eff}} \approx \frac{1}{2}\left[\frac{u(\boldsymbol{x}_i)}{\Delta u(\boldsymbol{x}_i)}\right]^2 \ ,$$

which turns out to be the same expression used with GUM uncertainties [1, G.4.2]. For example, if an estimate of $|\boldsymbol{\Gamma}|$ is considered to be reliable to about 10%, i.e. $|\boldsymbol{\Gamma}|/\Delta|\boldsymbol{\Gamma}| \approx 10$, then $\nu_{\text{eff}} = 50$ could be associated with the standard uncertainty $u(\boldsymbol{\Gamma})$.

8 Examples

Several examples involving the measurement of complex quantities, or where complex quantities influence a measurement result, are presented in this section. The scenarios are simple. It is intended only to show how the mathematical methods described in this part of the report can be applied, not to explore the many terms that might arise in the uncertainty budget for actual measurements.

8.1 Mismatch in power measurements

In a simple power measurement scenario, the measurement equation can be written as

$$P_{\mathrm{g}} = M P_{\mathrm{i}} \ , \tag{10}$$

where P_{i} is the net RF power available to a load with a reflection coefficient $\mathbf{\Gamma}_{\mathrm{s}}$, P_{g} is the power that the generator could deliver to an ideal load and

$$M = |1 - \mathbf{\Gamma}_{\mathrm{s}}\mathbf{\Gamma}_{\mathrm{g}}|^2 \tag{11}$$

is commonly referred to as the mismatch error. Mismatch depends on the complex reflection coefficients of the signal generator output, $\mathbf{\Gamma}_{\mathrm{g}}$, and $\mathbf{\Gamma}_{\mathrm{s}}$. Assuming that the phases of $\mathbf{\Gamma}_{\mathrm{g}}$ and $\mathbf{\Gamma}_{\mathrm{s}}$ are unknown, a type-B uncertainty should be associated with the estimate $M \approx 1$.

We begin by expressing the product as a single quantity

$$\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{s}}\mathbf{\Gamma}_{\mathrm{g}}$$

and associate standard uncertainties $u(\mathbf{\Gamma}_{\mathrm{g}})$ and $u(\mathbf{\Gamma}_{\mathrm{s}})$ with estimates of $\mathbf{\Gamma}_{\mathrm{g}}$ and $\mathbf{\Gamma}_{\mathrm{s}}$. Then, from the product rule of §5.1.4, the standard uncertainty associated with $\mathbf{\Gamma}$ is

$$u(\mathbf{\Gamma}) = \sqrt{2}u(\mathbf{\Gamma}_{\mathrm{s}})u(\mathbf{\Gamma}_{\mathrm{g}}) \ .$$

The mismatch

$$M = |1 - \mathbf{\Gamma}|^2 \tag{12}$$

is a real quantity, so the remainder of the analysis needs to use the GUM LPU to propagate uncertainty.

Assuming that M and P_{i} are independent, the LPU gives

$$\begin{aligned} u(P_{\mathrm{g}})^2 &= \left[\frac{\partial P_{\mathrm{g}}}{\partial M}\, u(M)\right]^2 + \left[\frac{\partial P_{\mathrm{g}}}{\partial P_{\mathrm{i}}}\, u(P_{\mathrm{i}})\right]^2 \\ &= [P_{\mathrm{i}}\, u(M)]^2 + [M\, u(P_{\mathrm{i}})]^2 \ . \end{aligned}$$

An expression for $u(M)$ can be obtained by rewriting equation (12) in terms of real and imaginary components

$$M = 1 - 2\Gamma_{\mathrm{re}} + \Gamma_{\mathrm{re}}^2 + \Gamma_{\mathrm{im}}^2 \ .$$

Then, differentiating with respect to each component, and remembering that $\mathbf{\Gamma} \approx 0$, we find just one non-zero sensitivity coefficient

$$\frac{\partial M}{\partial \Gamma_{\mathrm{re}}} = 2(\Gamma_{\mathrm{re}} - 1) \approx -2$$
$$\frac{\partial M}{\partial \Gamma_{\mathrm{im}}} = 2\Gamma_{\mathrm{im}} \approx 0 \ .$$

So,

$$u(M) = 2u(\Gamma_{\mathrm{re}}) = 2u(\mathbf{\Gamma})$$

and

$$u(P_{\mathrm{g}})^2 = [P_{\mathrm{i}}\, u(M)]^2 + [M\, u(P_{\mathrm{i}})]^2 \ .$$

The standard uncertainty $u(\mathbf{\Gamma})$ depends on the information available about $\mathbf{\Gamma}_{\mathrm{g}}$ and $\mathbf{\Gamma}_{\mathrm{s}}$ (see §5.1). When both magnitudes are known,

$$u(M) = \sqrt{2}\, |\mathbf{\Gamma}_{\mathrm{s}}||\mathbf{\Gamma}_{\mathrm{g}}| \ .$$

However, if one magnitude is known and the other is bounded the uncertainty is reduced

$$u(M) = |\mathbf{\Gamma}_{\mathrm{s}}||\mathbf{\Gamma}_{\mathrm{g}}|$$

and when both magnitudes are bounded it is further reduced

$$u(M) = \frac{|\mathbf{\Gamma}_{\mathrm{s}}||\mathbf{\Gamma}_{\mathrm{g}}|}{\sqrt{2}} \ .$$

Numerical example: To complete the uncertainty calculation, some numerical information about P_{i}, $\mathbf{\Gamma}_{\mathrm{g}}$ and $\mathbf{\Gamma}_{\mathrm{s}}$ is provided in Table 2.

Table 2: Numerical data for a simple power measurement

Quantity	Value	Uncertainty
$\lvert\mathbf{\Gamma}_{\mathrm{g}}\rvert$	< 0.310	-
$\lvert\mathbf{\Gamma}_{\mathrm{s}}\rvert$	< 0.083	-
P_i	$100\,\mu\mathrm{W}$	1%

In this case, there are upper bounds on the two reflection coefficients so $u(M) = 0.310 \times 0.083 \div \sqrt{2} = 0.018$. The combined standard uncertainty is

$$u(P_{\mathrm{g}}) = \sqrt{(0.018 \times 10^{-4})^2 + (10^{-6})^2} = 2.1 \times 10^{-6}\,\mathrm{W} \ .$$

Had we considered the magnitudes to be known (as is often done) the mismatch uncertainty is twice as big $u(M) = 0.310 \times 0.083 \times \sqrt{2} = 0.036$ and the combined standard uncertainty becomes

$$u(P_{\mathrm{g}}) = \sqrt{(0.036 \times 10^{-4})^2 + (10^{-6})^2} = 3.8 \times 10^{-6}\,\mathrm{W} \ .$$

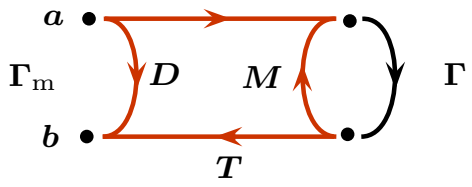

Figure 10: Residual error model of a one-port VNA measurement of $\mathbf{\Gamma}$. The value indicated by the VNA is $\mathbf{\Gamma}_\mathrm{m} = \boldsymbol{b}/\boldsymbol{a}$, although the quantity of interest is really $\mathbf{\Gamma}$. $\boldsymbol{D} \approx 0$, $\boldsymbol{M} \approx 0$ and $\boldsymbol{T} \approx 1$

.

8.2 One-port VNA measurements

The residual error model for a one-port VNA is shown in Figure 10, with the complex residual errors: $\boldsymbol{D}$, for residual directivity; $\boldsymbol{M}$, for residual source-match; and $\boldsymbol{T}$, for residual transmission-tracking.

We can express $\mathbf{\Gamma}$, in terms of the observed value $\mathbf{\Gamma}_\mathrm{m}$ and the residual errors, as

$$\mathbf{\Gamma} = \frac{\mathbf{\Gamma}_\mathrm{m} - \boldsymbol{D}}{\boldsymbol{M}(\mathbf{\Gamma}_\mathrm{m} - \boldsymbol{D}) + \boldsymbol{T}} \,. \tag{13}$$

The three complex partial derivatives of $\mathbf{\Gamma}$, with respect to residual directivity, source-match and tracking are

$$\frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{D}} = -\frac{\boldsymbol{T}}{[\boldsymbol{M}(\mathbf{\Gamma}_\mathrm{m} - \boldsymbol{D}) + \boldsymbol{T}]^2} \approx -1 \tag{14}$$

$$\frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{T}} = -\frac{\mathbf{\Gamma}_\mathrm{m} - \boldsymbol{D}}{[\boldsymbol{M}(\mathbf{\Gamma}_\mathrm{m} - \boldsymbol{D}) + \boldsymbol{T}]^2} \approx -\mathbf{\Gamma}_\mathrm{m} \tag{15}$$

$$\frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{M}} = -\frac{(\mathbf{\Gamma}_\mathrm{m} - \boldsymbol{D})^2}{[\boldsymbol{M}(\mathbf{\Gamma}_\mathrm{m} - \boldsymbol{D}) + \boldsymbol{T}]^2} \approx -\mathbf{\Gamma}_\mathrm{m}^2 \,. \tag{16}$$

The expressions shown on the right are a consequence of the usual approximations $\boldsymbol{D} \approx 0$, $\boldsymbol{M} \approx 0$ and $\boldsymbol{T} \approx 1$.

A European guidance document on the evaluation of VNA measurement uncertainty [14] describes methods of estimating the magnitude of the residuals. Estimates of $|\boldsymbol{D}|$, $|\boldsymbol{M}|$ and $|\boldsymbol{T} - 1|$ are considered as input uncertainties (i.e. $u(\boldsymbol{D}) \propto |\boldsymbol{M}|$, $u(\boldsymbol{M}) \propto |\boldsymbol{M}|$ and $u(\boldsymbol{T}) \propto |\boldsymbol{T} - 1|$) and the corresponding components of uncertainty in $\mathbf{\Gamma}$ are

$$u(\boldsymbol{D}), \quad |\mathbf{\Gamma}_\mathrm{m}|\, u(\boldsymbol{T}), \text{ and } \quad |\mathbf{\Gamma}_\mathrm{m}^2|\, u(\boldsymbol{M}) \,,$$

for directivity, reflection-tracking and source-match, respectively.

The combined standard uncertainty is then found from equation (8)

$$u(\mathbf{\Gamma}) = \left[u(\boldsymbol{D})^2 + (|\mathbf{\Gamma}_\mathrm{m}|\, u(\boldsymbol{T}))^2 + (|\mathbf{\Gamma}_\mathrm{m}^2|\, u(\boldsymbol{M}))^2 + R_\mathrm{VRC}^2\right]^{1/2} \,,$$

where R_VRC is introduced here as a catch-all term for uncertainty due to random measurement errors (see [14] and [15]).

Table 3: Numerical data for a simple power measurement

Quantity	Value	Uncertainty
$\boldsymbol{\Gamma}_{\mathrm{m}}$	0.08 -j 0.05	-
$\lvert\boldsymbol{D}\rvert$	0.01	-
$\lvert\boldsymbol{M}\rvert$	0.01	-
$\lvert\boldsymbol{T}\rvert$	0.995	-
R_{VRC}	0	0.006

Numerical example: The data in Table 3 would be typical of a one-port VNA measurement at low frequency. Associating each residual error term with a ring distribution, the corresponding terms in the uncertainty budget are

Term	Uncertainty
$u(\boldsymbol{D})$	7.1×10^{-3}
$\lvert\boldsymbol{\Gamma}_{\mathrm{m}}^2\rvert\, u(\boldsymbol{M})$	6.3×10^{-5}
$\lvert\boldsymbol{\Gamma}_{\mathrm{m}}\rvert\, u(\boldsymbol{T})$	3.3×10^{-4}
R_{VRC}	6×10^{-3}

The combined standard uncertainty is the root-sum-square of the data in the left column, $u(\boldsymbol{\Gamma}) = 0.0093$. If, instead of a ring, each residual is associated with a uniform disk distribution (i.e. taking the estimates as upper bounds on the actual errors), we obtain $u(\boldsymbol{\Gamma}) = 0.0078$.

8.3 Attenuation measurements

A simple attenuation measurement procedure takes the ratio of two power measurements, one made with a direct connection between a power meter and a signal source, the other with a device inserted between the source and meter.

There are four complex terms in the measurement equation. The reflection coefficients of the signal source and the power sensor, $\boldsymbol{\Gamma}_{\mathrm{g}}$ and $\boldsymbol{\Gamma}_{\mathrm{s}}$, respectively, as well as reflection coefficients associated with the entrance and exit ports of the device under test (DUT), denoted $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{22}$ (see Figure 11).

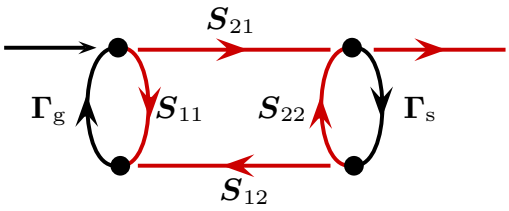

Figure 11: Signal flow diagram for the transmission measurement with a DUT connected between a source and power meter.

The magnitude squared of the transmission coefficient can be expressed as

$$S_{21}^2 = R \left| \frac{1 - \boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}}{(1 - \boldsymbol{S}_{11}\boldsymbol{\Gamma}_{\mathrm{g}})(1 - \boldsymbol{S}_{22}\boldsymbol{\Gamma}_{\mathrm{s}}) - S_{21}^2\boldsymbol{\Gamma}_{\mathrm{s}}\boldsymbol{\Gamma}_{\mathrm{g}}} \right|^2 ,$$

where R is a measured power ratio and $\boldsymbol{\Gamma}_{\mathrm{g}}$, $\boldsymbol{\Gamma}_{\mathrm{s}}$, $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{22}$ are all complex reflection coefficients with values all close to zero.

We now show how the ratio of complex terms

$$1 + \boldsymbol{\Psi} = \frac{1 - \boldsymbol{\Gamma}_\mathrm{s}\boldsymbol{\Gamma}_\mathrm{g}}{(1 - \boldsymbol{S}_{11}\boldsymbol{\Gamma}_\mathrm{g})(1 - \boldsymbol{S}_{22}\boldsymbol{\Gamma}_\mathrm{s}) - S_{21}^2\boldsymbol{\Gamma}_\mathrm{s}\boldsymbol{\Gamma}_\mathrm{g}}$$

can be handled by the rules in §6.1.1.

First, consider the denominator. Expanding the product in parentheses, dropping the term $\boldsymbol{S}_{11}\boldsymbol{S}_{22}\boldsymbol{\Gamma}_\mathrm{g}\boldsymbol{\Gamma}_\mathrm{s}$ which is small, and approximating S_{21}^2 by the nominal value $S_{\mathrm{n}\cdot 21}^2$, we obtain

$$1 - \boldsymbol{S}_{11}\boldsymbol{\Gamma}_\mathrm{g} - \boldsymbol{S}_{22}\boldsymbol{\Gamma}_\mathrm{s} - S_{\mathrm{n}\cdot 21}^2\boldsymbol{\Gamma}_\mathrm{s}\boldsymbol{\Gamma}_\mathrm{g} \ .$$

writing

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{S}_{11}\boldsymbol{\Gamma}_\mathrm{g} \\ \boldsymbol{b} &= \boldsymbol{S}_{22}\boldsymbol{\Gamma}_\mathrm{s} \\ \boldsymbol{c} &= S_{\mathrm{n}\cdot 21}^2\boldsymbol{\Gamma}_\mathrm{s}\boldsymbol{\Gamma}_\mathrm{g}\end{aligned}$$

the standard uncertainty associated with the denominator (using the rule for addition in §6.1.1) is

$$\sqrt{u(\boldsymbol{a})^2 + u(\boldsymbol{b})^2 + u(\boldsymbol{c})^2} \ .$$

For the numerator, we write

$$\boldsymbol{d} = \boldsymbol{\Gamma}_\mathrm{s}\boldsymbol{\Gamma}_\mathrm{g} \ ,$$

so $u(\boldsymbol{d})$ is the associated standard uncertainty.

Now, the best estimates of the numerator and denominator are each unity, so the uncertainty of the ratio is simply[11]

$$u(\boldsymbol{\Psi}) = \sqrt{u(\boldsymbol{a})^2 + u(\boldsymbol{b})^2 + u(\boldsymbol{c})^2 + u(\boldsymbol{d})^2} \ .$$

Attenuation, the quantity indended to be measured, is defined as

$$A = -10 \ \log_{10} S_{21}^2$$

so

$$A = -10 \ \log_{10} R - 10 \ \log_{10} |1 + \boldsymbol{\Psi}|^2 \ .$$

We estimate $\boldsymbol{\Psi} \approx 0$ and $u(\boldsymbol{\Psi}) \ll 1$, so

$$|1 + \boldsymbol{\Psi}|^2 \approx 1 + 2\mathrm{Re}(\boldsymbol{\Psi}) \ .$$

Finally, since $\log_e(1 + 2x) \approx 2x$, for small x, we obtain

$$\begin{aligned}A &\approx -10 \ \log_{10} R - 2\frac{10}{\log_e 10}\mathrm{Re}(\boldsymbol{\Psi}) \\ &\approx -10 \ \log_{10} R - 8.686\,\mathrm{Re}(\boldsymbol{\Psi}) \ .\end{aligned}$$

Since A is real-valued, the remaining steps use the GUM LPU for uncertainty propagation. The standard uncertainty associated with the attenuation measurement is

$$\begin{aligned}u(A)^2 &= 8.686^2 \, u(\boldsymbol{\Psi})^2 + u(N_\mathrm{dB})^2 \\ &= 8.686^2 \, [u(\boldsymbol{a})^2 + u(\boldsymbol{b})^2 + u(\boldsymbol{c})^2 + u(\boldsymbol{d})^2] + u(N_\mathrm{dB})^2 \ ,\end{aligned}$$

[11] Here, the rule in §6.1.1 for multiplication and division is used. However, because the denominator in every term is unity, it appears as though the root-sum-square rule for addition and subtraction is being used.

where $u(N_{\mathrm{dB}})^2$ has been introduced as a catch-all term for uncertainty due to random measurement errors, expressed in dB.

Each of $u(\boldsymbol{a})$, $u(\boldsymbol{b})$, $u(\boldsymbol{c})$ and $u(\boldsymbol{d})$ are related to information about $\boldsymbol{\Gamma}_{\mathrm{g}}$, $\boldsymbol{\Gamma}_{\mathrm{s}}$, $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{22}$. If all magnitudes are known

$$
\begin{aligned}
u(\boldsymbol{a}) &= \frac{|\boldsymbol{S}_{11}||\boldsymbol{\Gamma}_{\mathrm{g}}|}{\sqrt{2}} \\
u(\boldsymbol{b}) &= \frac{|\boldsymbol{S}_{22}||\boldsymbol{\Gamma}_{\mathrm{s}}|}{\sqrt{2}} \\
u(\boldsymbol{c}) &= S^2_{\mathrm{n}\cdot 21}\frac{|\boldsymbol{\Gamma}_{\mathrm{s}}||\boldsymbol{\Gamma}_{\mathrm{g}}|}{\sqrt{2}} \\
u(\boldsymbol{d}) &= \frac{|\boldsymbol{\Gamma}_{\mathrm{s}}||\boldsymbol{\Gamma}_{\mathrm{g}}|}{\sqrt{2}} \ .
\end{aligned}
$$

This result is essentially equivalent to one obtained by Harris and Warner [16, §3.2]. However, a reduction in the uncertainty will be obtained if different type-B uncertainties are associated with $u(\boldsymbol{a})$, $u(\boldsymbol{b})$, $u(\boldsymbol{c})$ and $u(\boldsymbol{d})$ (see §5.1).

9 Concluding comments

The RF and microwave metrology community has developed full bivariate extensions to GUM methods to deal with complex quantities. This work is largely accomplished, however, the computations can be daunting and these full methods have not been widely adopted.

The simpler practice of reporting complex measurement uncertainty as a circle in the complex plane originated at NPL in 1992 and the associated uncertainty propagation formulae, equations (7) and (8) in §6.1, were developed later at the Swedish National Testing and Research Institute [10]. These ideas underpin a full gamut of extensions to the GUM that enable complex measurement uncertainty calculations to be handled in an objective and standardised manner, without too much laborious mathematics. These methods offer a practical alternative to the more rigorous bivariate approach.

The RF and microwave metrology community now needs to focus on points of difference between the real-valued framework for reporting uncertainty in the GUM and the more complicated framework needed for complex-valued problems. Differences arise when multivariate concepts replace univariate ones, such as: a region of uncertainty that depends on several parameters and replaces the notion of an uncertainty interval; a component of uncertainty matrix, with four elements, that replaces a real-valued component of uncertainty. These multivariate concepts will be covered in Part 2, but to some extent they are already present in the concepts introduced in this document. Indeed, some quantities used here are effective summary values for their multivariate counterparts and offer better intuitive understanding of uncertainty calculation results.

To conclude, the two-dimensional character of complex measurement uncertainty needs to be recognised. This is an important and necessary step towards an appreciation of the bivariate mathematical problem, and offers insight into the quantities involved. Hence, the importance of describing the uncertainty circle of an estimate in the complex plane. At the same time, methods for evaluation and propagation of uncertainty should be simple and standardised to facilitate uptake by the wider measurement community. These goals are achieved by the methods presented in this part of the report.

10 Changes and corrections

Section 3.1 A footnote has been added, stating that the coverage factor $k_{2,p}$ is usually applied to an ellipse (not a circle) in the conventional statistics treatment of type-A uncertainty. Its use with a circular uncertainty region is an approximation that generally leads to conservative uncertainty statements.

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