The uncertainty of a complex quantity with unknown phase

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Introduction

ANAMET-29 : (March 2008)

- cases with no phase information
- view uncertainty in the complex plane
- uniform ring and disk distributions
- product of terms with no phase information
- mismatch uncertainty (improved)

UKAS-M3003 : (another case)

- no phase information *and* uncertainty in magnitude

Validity :

- do these uncertainty models work?
- how do they perform?

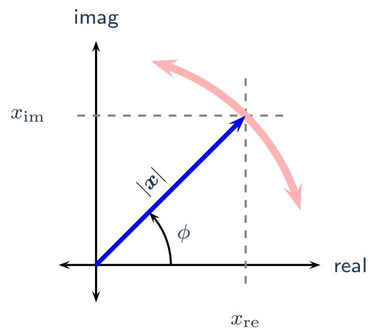
Notes

- This talk expands on material presented to the 29$^{\text{th}}$ ANAMET meeting (March 2008). That talk, entitled "Mismatch uncertainty: representations for complex calculations", discussed expressions of uncertainty for a complex quantity when there is complete ignorance of phase.
- This talk revises some of the material given in 2008 and discusses another case used in the UKAS Guide M3003.
- Some performance-testing of the proposed models of uncertainty is presented.

Complex uncertainty when phase is unknown

A complex quantity:

- The likely value is evenly distributed around the origin



- The uncertainty is characterised by magnitude alone
- Simple relation between complex uncertainty and real and imaginary component uncertainties

Several cases:

- The magnitude is known
- The magnitude is bounded above
- The magnitude is uncertain (c.f. M3003)
- The magnitude a product of quantities with unknown phase

Notes

The situation of interest is when the phase of a complex quantity is unknown and there is some information about its magnitude.

This situation occurs frequently in RF measurement situations but is difficult to handle formally.

In the expressions obtained, there is a simple relation between the uncertainty in the complex plane and the uncertainty in the real and imaginary components of the quantity.

Magnitude is known

- possible values of $\boldsymbol{x}$ form a uniform ring around the origin

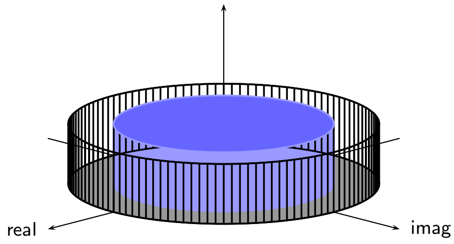
$$\boxed{|\boldsymbol{x}| = a}$$

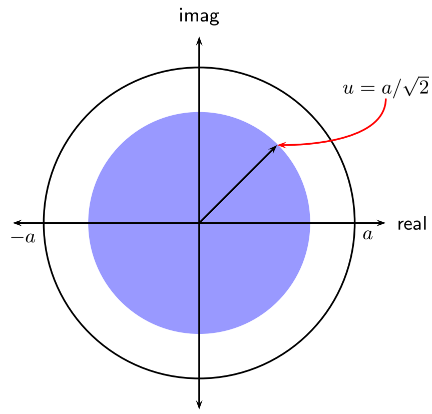
- covariance matrix

$$\mathbf{v}(\boldsymbol{x}) = \frac{a^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- uncertainty matrix

$$\mathbf{u}(\boldsymbol{x}) = \frac{a}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$





Reference: B. D. Hall, Metrologia **44** (2007) L62–L67

Notes

The information available puts $\boldsymbol{x}$ somewhere on a circle.

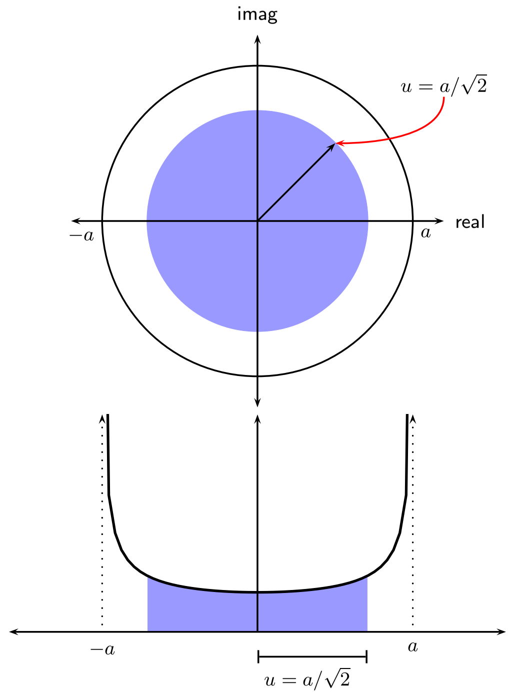
Note that a typical coverage factor will magnify the uncertainty region to encompass the ring.

In practice, this type-B distribution (and those following) will be used to propagate uncertainty information about a number of influence quantities. We need to convert range information (the circle diameter, in this case) into the standard uncertainties (or standard variances) required by propagation algorithms.

The arcsine distribution

The ring distribution is related to the arcsine (U-shaped) distribution.

- The arcsine distribution has long been associated with mismatch
- It is the marginal distribution of the ring distribution



Reference: I. A. Harris and F. L. Warner, IEE Proc. **128** Pt. H, No. 1, (1981) 35–41

Notes

In the analysis of RF measurements, expressions arise in the form $|1+\boldsymbol{\epsilon}|^2$, where $\boldsymbol{\epsilon}$ is a complex quantity with unknown phase. We may write ($|\boldsymbol{\epsilon}|^2 \ll 1$)

$$\begin{aligned} |1+\boldsymbol{\epsilon}|^2 &= (1+\boldsymbol{\epsilon})(1+\boldsymbol{\epsilon}^*) \\ &= 1 + 2\mathrm{Re}\{\boldsymbol{\epsilon}\} + |\boldsymbol{\epsilon}|^2 \\ &\approx 1 + 2\mathrm{Re}\{\boldsymbol{\epsilon}\} \qquad (|\boldsymbol{\epsilon}|^2 \ll 1) \end{aligned}$$

The projection (the marginal distribution) of $\boldsymbol{\epsilon}$ along the real axis is of interest in this case.

A related expression is

$$|1+\boldsymbol{\epsilon}| \approx 1 + \mathrm{Re}\{\boldsymbol{\epsilon}\} \qquad (|\boldsymbol{\epsilon}| \ll 1)$$

Magnitude is bounded

- possible values form a uniform disk around the origin

$$\boxed{|\boldsymbol{x}| <= a}$$

- covariance matrix

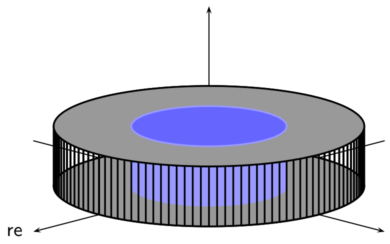
$$\mathbf{v}(\boldsymbol{x}) = \frac{a^2}{4}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

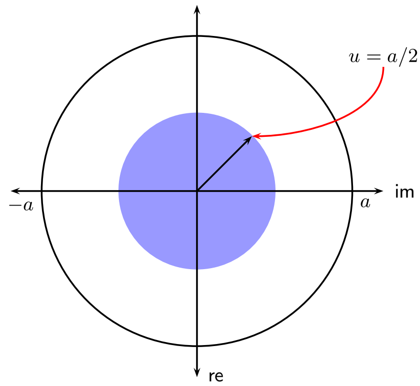
- uncertainty matrix

$$\mathbf{u}(\boldsymbol{x}) = \frac{a}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

- equivalent-ring radius

$$a' = a/\sqrt{2}$$





Reference: B. D. Hall, Metrologia **44** (2007) L62–L67

Notes

Although RF component specifications are often reported in this way, uncertainty guides have not treated this case.

The uncertainty associated with the disk distribution is less than a ring (which, intuitively, is more precisely defined), because the disk distribution allows for errors close to the centre, while the ring distribution does not.

If the ring diameter (magnitude) is actually *measured* then it is likely to be within the bounded range of the uniform disk (which is likely to be a type-specification).

The uncertainty of a product

- Quantity of interest

$$\boxed{\mathbf{\Gamma} = \mathbf{\Gamma}_\mathrm{s}\mathbf{\Gamma}_\mathrm{g}}$$

- The covariance matrices associated with $\mathbf{\Gamma}_\mathrm{s}$ and $\mathbf{\Gamma}_\mathrm{g}$ are:

$$\mathbf{v}(\mathbf{\Gamma}_\mathrm{g}) = u_\mathrm{g}^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}(\mathbf{\Gamma}_\mathrm{s}) = u_\mathrm{s}^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The covariance of the product is

$$\mathbf{v}(\mathbf{\Gamma}) = 2(u_\mathrm{s} u_\mathrm{g})^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The uncertainty matrix is

$$\mathbf{u}(\mathbf{\Gamma}) = \sqrt{2}\, u_\mathrm{s} u_\mathrm{g} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- equivalent-ring radius

$$a' = 2\, u_\mathrm{s} u_\mathrm{g}$$

Note

Consider two random variables associated with the complex quantities[a]

$$\begin{aligned} \boldsymbol{z}_1 &= x_1 + \mathrm{i} y_1 \\ \boldsymbol{z}_2 &= x_2 + \mathrm{i} y_2 \ . \end{aligned}$$

The real and imaginary components of these random variables have means of zero. The variances of the real and imaginary components of $\boldsymbol{z}_1$ are both equal to σ_1^2, similarly the variances of the real and imaginary components of $\boldsymbol{z}_2$ are equal to σ_2^2. The covariance between components is zero.

The expectation of the product is zero, i.e.,

$$\boldsymbol{z}_1 \boldsymbol{z}_2 = (x_1 x_2 - y_1 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1)$$

and the expectation

$$\begin{aligned} E(x_1 x_2 - y_1 y_2) &= E(x_1 x_2) - E(y_1 y_2) \\ &= E(x_1)E(x_2) - E(y_1)E(y_2) \\ &= 0 \end{aligned}$$

similarly $E(x_1 y_2 + x_2 y_1) = 0$.

[a] The author is grateful to R. Willink for this derivation.

The variances and covariance are then obtained as follows:

$$\begin{aligned} & E\left[(x_1 x_2 - y_1 y_2)^2\right] \\ &\quad = E\left[x_1^2 x_2^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2\right] \\ &\quad = E(x_1^2 x_2^2) - 0 + E(y_1^2 y_2^2) \\ &\quad = \sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_2^2 \\ &\quad = 2\sigma_1^2 \sigma_2^2 \ , \end{aligned}$$

similarly

$$E\left[(x_1 y_2 + x_2 y_1)^2\right] = 2\sigma_1^2 \sigma_2^2$$

and

$$\begin{aligned} & E\left[(x_1 x_2 - y_1 y_2)(x_1 y_2 + x_2 y_1)\right] \\ &\quad = E(x_1^2 x_2 y_2) - E(y_2^2 x_1 y_1) + E(x_2^2 x_1 y_1) \\ &\qquad - E(y_1^2 x_2 y_2) \\ &\quad = 0 \end{aligned}$$

Uncertain magnitude

A magnitude may be reported with uncertainty, i.e.:

$$|\boldsymbol{x}| = a \pm u(a)$$

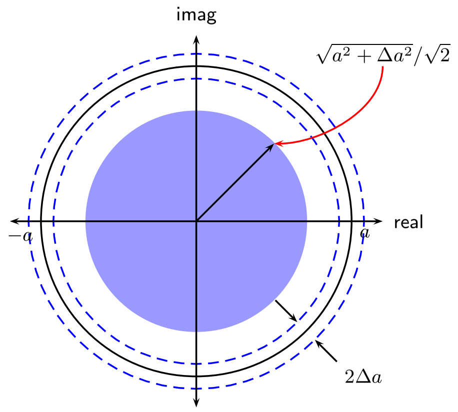
This can be thought of as

- an annulus with radii $a \pm \Delta a$
- a covariance matrix

$$\mathbf{v}(\boldsymbol{x}) = \frac{a^2 + \Delta a^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- an uncertainty matrix

$$\mathbf{u}(\boldsymbol{x}) = \sqrt{\frac{a^2 + \Delta a^2}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



But how is Δa related to $u(a)$?

Notes

When there is a uniform annular distribution, with an inner diameter r_1 and an outer diameter r_2, the covariance matrix is

$$\mathbf{v} = \frac{1}{4} \begin{bmatrix} r_1^2 + r_2^2 & 0 \\ 0 & r_1^2 + r_2^2 \end{bmatrix}$$

Writing $r_1 = a - \Delta a$ and $r_2 = a + \Delta a$ we can express the diagonal elements

$$\begin{aligned} r_1^2 + r_2^2 &= (a - \Delta a)^2 + (a + \Delta a)^2 \\ &= (a^2 - 2a\Delta a + \Delta a^2) + (a^2 + 2a\Delta a + \Delta a^2) \\ &= 2(a^2 + \Delta a^2) \end{aligned}$$

and so

$$\mathbf{v} = \frac{a^2 + \Delta a^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

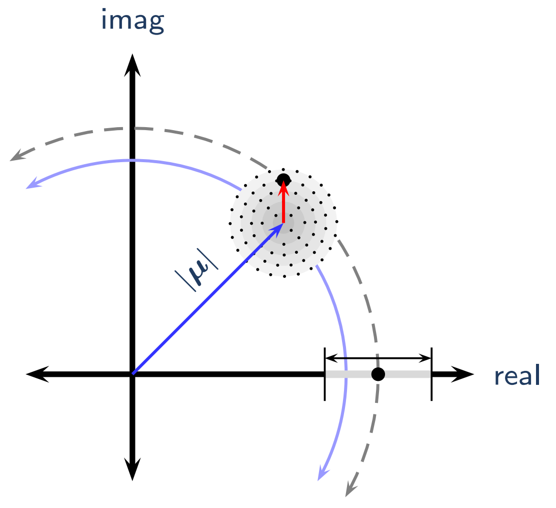
Note that this is in the same form as a uniform ring distribution with radius $\sqrt{a^2 + \Delta a^2}$.

So the marginal distribution of a uniform annulus is equivalent to an arcsine distribution with a slightly larger ring-radius.

An error model for uncertain magnitude

- A complex quantity $\boldsymbol{\mu}$ is of interest
- On each independent experiment, the orientation of $\boldsymbol{\mu}$ changes.
- Observations are subject to error
- Only magnitude is measured

- The magnitude of $\boldsymbol{\mu}$ is fixed
- The orientations of $\boldsymbol{\mu}$ are uniformly distributed
- The error around $\boldsymbol{\mu}$ is normally distributed (2-D) with variance σ^2



The uncertainty matrix associated with independent observations of $|\boldsymbol{\mu}|$ is

$$\mathbf{u}(\boldsymbol{\mu}) = \sqrt{\frac{|\boldsymbol{\mu}|^2 + 2\sigma^2}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notes

Consider a complex quantity of interest $\boldsymbol{\mu}$.

A procedure obtains observations $\boldsymbol{X}$ of $\boldsymbol{\mu}$ and is subject to an unbiased additive error. The variance of observations about the measurand is

$$E[(\boldsymbol{X} - \boldsymbol{\mu})^2] = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} .$$

The variance of observations about the origin is

$$E[\boldsymbol{X}^2] = \begin{bmatrix} \mu_r^2 + \sigma^2 & \mu_r \mu_i \\ \mu_r \mu_i & \mu_i^2 + \sigma^2 \end{bmatrix} ,$$

where $\mu_r = |\boldsymbol{\mu}| \cos\phi$ is the real component of $\boldsymbol{\mu}$ and $\mu_i = |\boldsymbol{\mu}| \sin\phi$ is the imaginary component.

We further suppose that $\boldsymbol{\mu}$ is randomly oriented at each observation. That is, ϕ is an independent uniformly distributed random variable.

The variance of this 'mixture' is now the average of $E[\boldsymbol{X}^2]$ over $\phi \in [0, 2\pi]$.

We obtain

$$E[\boldsymbol{X}^2_{\text{mix}}] = \begin{bmatrix} \frac{1}{2}|\boldsymbol{\mu}|^2 + \sigma^2 & 0 \\ 0 & \frac{1}{2}|\boldsymbol{\mu}|^2 + \sigma^2 \end{bmatrix}$$
$$= \frac{|\boldsymbol{\mu}|^2 + 2\sigma^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Note that as $|\boldsymbol{\mu}| \to 0$ this yields the simple additive error variance as expected ($E[(\boldsymbol{X} - \boldsymbol{\mu})^2]$). Conversely, if $\sigma \to 0$ we obtain the variance for a uniform ring.

Summary of cases

Single quantity:

Parameter	Distribution	Marginal Standard Uncertainty	Equivalent-ring radius
$\lvert\boldsymbol{x}\rvert = a$	Ring	$a/\sqrt{2}$	a
$\lvert\boldsymbol{x}\rvert \leq a$	Disk	$a/2$	$a/\sqrt{2}$
$\lvert\boldsymbol{x}\rvert = a \pm u(a)$	Annulus	$\sqrt{a^2 + 2u^2(a)}/\sqrt{2}$	$\sqrt{a^2 + 2u^2(a)}$

Product of two quantities:

- The characteristic uncertainty is $\sqrt{2}$ times the product of standard uncertainties

$$u(\boldsymbol{\Gamma}) = \sqrt{2}\, u(\boldsymbol{\Gamma}_\mathrm{a}) u(\boldsymbol{\Gamma}_\mathrm{b})$$

- There will be different possible combinations (ring-ring, ring-disk, etc)
- Equivalent-ring radius

$$a' = 2\, u(\boldsymbol{\Gamma}_\mathrm{a}) u(\boldsymbol{\Gamma}_\mathrm{b})$$

Notes

In all cases, the covariance matrices have the same diagonal form, which is characterised by just one parameter – the standard variance of the real and imaginary components.

The marginal standard uncertainties can all be treated as U-shaped distributions with an appropriately scaled ring-radius.

Checking the performance of uncertainty models

Simple simulations:

- Use a simple error model to simulate experimental observations
- express observations as a function of experimental errors
- use random number generators to simulate errors

Data processing:

- Apply uncertainty analysis as if these were actual measurements
- estimate the measurand as a function of observations and uncertain influences
- obtain the measurement uncertainty by GUM methods

Check for success:

- For each simulated experiment, check if the measurand lies within the uncertainty interval
- count successes

Notes

In what follows, we use random number generators to simulate measurement errors with appropriate distributions (uniform ring, uniform disk, uncertain-magnitude ring, etc).

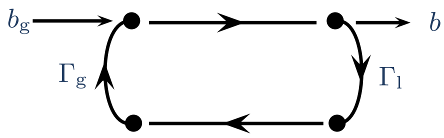
We combine error simulation with simple measurement models to obtain large data-sets representing many independent observations of the quantity of interest.

Ddata processing is applied to the simulated observations. We assume limited knowledge about the actual errors, as would be the case in real experiments. We use GUM methods and the uncertainties obtained earlier to evaluate the measurement uncertainty of each simulated measurement. The results can be compared with the actual quantity value used in the simulation.

In this way, the success-rate of the GUM uncertainty calculation using these models of uncertainty can be evaluated.

Power measurements

Measurement model:



- simulate observations:
  - p_{gen} is the (fixed) measurand
  - observe $p_i = p_{\mathrm{gen}}/m_i + n_i$
  - $n_i \sim N(0, \sigma^2)$
  - $m_i = |1 - \mathbf{\Gamma}_i|^2$
  - $\mathbf{\Gamma}_i \sim$ ring(a), or other 2-D uniform distribution
- data processing (value):

$$p_{\mathrm{gen}\cdot i} \approx p_i \quad (n \approx 0, m \approx 1)$$

- data processing (uncertainty):

$$u(p_{\mathrm{gen}\cdot i}) = \sqrt{[p_i\, u(m)]^2 + u(n)^2},\ u(n) = \sigma$$

Notes

The full measurement equation is

$$p_{\mathrm{gen}} = m\,(p_i - n)\ ,$$

where m and n represent the mismatch and noise errors respectively (about which we have limited knowledge, $m \approx 1$ and $n \approx 0$.).

The sensitivity coefficients are

$$\frac{\partial p_{\mathrm{gen}}}{\partial m} = p_i \quad \text{and} \quad \frac{\partial p_{\mathrm{gen}}}{\partial n} = -m\ .$$

So the components of uncertainty are $p_i\, u(m)$ and $u(n)$ for mismatch and noise, respectively.

Power - mismatch only

First, ignore noise ($\sigma = 0$).

Distribution type	Mismatch std. uncertainty	success-rate (%)
ring × ring	0.1414	100.0
disk × ring	0.1000	97.0
disk × disk	0.0707	94.8
u-ring × u-ring	0.1561	100.0

Note:

- Disk radius $a = 0.1$ and $u(a) = a/3$ for the uncertain ring
- Nominal coverage is 95 %
- U-shaped distribution (ring case) is conservative
- The number of simulations $N = 10^5$
- Uncertainty in success-rate $\approx 0.1\%$ (1-sigma)

Notes

When the U-shaped error distribution is the dominant source of error (ring × ring case), the distribution is not at all Gaussian-like, so the GUM method can be expected to struggle. However, we see that this leads to conservative uncertainty statements.

The disk × disk case has an error distribution that is much more Gaussian-like and the performance observed comes close to nominal.

Power - mismatch and noise

In this case, noise is added.

Observed success rates (in %):

Distribution type	Mismatch uncertainty	σ 0.01	0.03	0.1	0.3	1.0
ring × ring	0.1414	100.0	99.4	96.2	95.5	95.9
disk × ring	0.1000	98.7	97.4	95.4	95.4	95.5
disk × disk	0.0707	95.0	95.0	95.1	95.2	95.2
u-ring × u-ring	0.1561	100.0	99.9	97.0	95.9	96.1

Note:

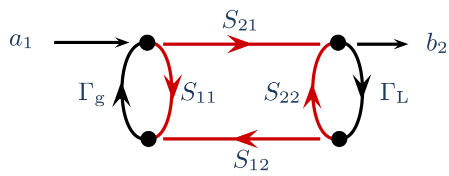
- Disk radius $a = 0.1$ and $u(a) = a/3$ for the uncertain ring
- Nominal coverage is 95 %
- The number of simulations $N = 10^5$
- Uncertainty in success-rate $\approx 0.1\%$ (1-sigma)
- The disk × disk case already performed well without noise

Notes

When a second component of uncertainty is present, the conservative effect of the U-shaped distribution can be mitigated. We see that this happens once the noise error is at least as large as the mismatch error.

Transmission measurements

Measurement model:



- simulated measurements:

$$\text{ratio}_i = |\boldsymbol{S}_{21}|^2 \frac{|1 - [\boldsymbol{\Gamma}_g \boldsymbol{\Gamma}_l]_i|^2}{|(1 - [\boldsymbol{S}_{11}\boldsymbol{\Gamma}_g]_i)(1 - [\boldsymbol{S}_{22}\boldsymbol{\Gamma}_l]_i) - \boldsymbol{S}_{12}\boldsymbol{S}_{21}[\boldsymbol{\Gamma}_g\boldsymbol{\Gamma}_l]_i|^2} + n_i$$

  each term $[\cdots]_i \sim$ ring(a), or other 2-D uniform distribution

- data processing (value):

$$|\boldsymbol{S}_{21\cdot i}|^2 \approx \text{ratio}_i$$

- data processing (uncertainty):

$$u(|\boldsymbol{S}_{21\cdot i}|^2) = \sqrt{u_1^2 + u_2^2 + u_3^2 + |\boldsymbol{S}_{12}\boldsymbol{S}_{21}|^2 u_4^2 + \sigma^2}$$

  where u_1, u_2 , etc, are standard uncertainties associated with the marginal distributions of $|\boldsymbol{S}_{11}\boldsymbol{\Gamma}_g|^2$, $|\boldsymbol{S}_{22}\boldsymbol{\Gamma}_l|^2$, etc.

Notes

The measurement equation is

$$|\boldsymbol{S}_{21\cdot i}|^2 = \left|\frac{m2}{m1}\right|^2 (\text{ratio}_i - n) \,,$$

where

$$m1 = 1 - \boldsymbol{\Gamma}_1$$

and

$$m2 = (1 - \boldsymbol{\Gamma}_2)(1 - \boldsymbol{\Gamma}_3) - \boldsymbol{S}_{12}\boldsymbol{S}_{21}\boldsymbol{\Gamma}_4 \,.$$

The terms $\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$, $\boldsymbol{\Gamma}_3$ and $\boldsymbol{\Gamma}_2$ are each the product of a pair complex quantities. They are treated as independent errors.

Assume that all $|\boldsymbol{\Gamma}_i| \ll 1$, so the mismatch ratio is approximately

$$1 + 2\text{Re}(\boldsymbol{\Gamma}_1) - 2\text{Re}(\boldsymbol{\Gamma}_2) - 2\text{Re}(\boldsymbol{\Gamma}_3) - 2|\boldsymbol{S}_{12}\boldsymbol{S}_{21}|\text{Re}(\boldsymbol{\Gamma}_4)$$

The uncertainties associated with the real components are simply the marginal uncertainties of the various complex distributions. So we write

$$u_1 = u(2\text{Re}(\boldsymbol{\Gamma}_1)) \,, \text{ etc}.$$

Transmission - mismatch only

First, ignore noise ($\sigma = 0$)

Distribution type	Mismatch uncertainty	success-rate (%)
ring × ring	0.1414	93.9
disk × ring	0.1000	94.9
disk × disk	0.0707	95.1
u-ring × u-ring	0.1561	95.3

Note:

- Disk radius $a = 0.1$ and $u(a) = a/3$ for the uncertain ring
- Nominal coverage is 95 %
- The number of simulations $N = 10^5$
- Uncertainty in success-rate $\approx 0.1\%$ (1-sigma)
- $|\boldsymbol{S}_{12}\boldsymbol{S}_{21}| = 10^{-4}$

Notes

There is no single dominant error in these measurements and we see that performance is close to nominal in all cases.

Nevertheless, the ring × ring case is slightly optimistic, presumably due to the non-Gaussian nature of the U-shaped errors. This is a different behaviour from the simple power measurement and is probably due to the fact that the errors now contribute to both the numerator and denominator of a mismatch ratio.

Transmission - mismatch and noise

In this case, noise is added (relative to $|\boldsymbol{S}_{12}\boldsymbol{S}_{21}|$)

Observed success rates (in %):

Distribution type	Mismatch uncertainty	$\sigma/\lvert\boldsymbol{S}_{12}\boldsymbol{S}_{21}\rvert$ 0.01	0.03	0.1	0.3	1.0
ring × ring	0.1414	94.4	94.2	94.1	95.4	97.0
disk × ring	0.1000	94.6	94.7	94.8	95.3	95.8
disk × disk	0.0707	95.3	94.6	94.8	95.4	95.6
u-ring × u-ring	0.1561	95.8	95.5	95.9	95.5	97.6

Note:

- Disk radius $a = 0.1$ and $u(a) = a/3$ for the uncertain ring
- Nominal coverage is 95 %
- The number of simulations $N = 10^4$
- Uncertainty in success-rate $\approx 0.2\%$ (1-sigma)
- $|\boldsymbol{S}_{12}\boldsymbol{S}_{21}| = 10^{-4}$

Notes

The addition of noise raises the performance of the ring × ring and u-ring × u-ring cases when the noise level is similar in magnitude to mismatch. What is notable here is that when noise dominates the success-rate becomes conservative ($\approx 97.5\%$).

This needs further investigation. There is probably correlation between the magnitude of the observed result and the uncertainty interval evaluated in that case, effectively leading to a one-sided uncertainty interval. The way noise is introduced in the model should be reconsidered.

Conclusions

- The uniform ring (corresponding to the U-shaped or arcsine 1-D distributions) is generally a conservative statement of uncertainty – a sort of 'worst-case' model.
- The disk and disk-product distributions yield lower uncertainties (than the ring) while accurately dealing with the underlying measurement error model.
- The uncertain-ring distribution is slightly more conservative than the uniform ring, but the effect on coverage is quite small (the difference between the M3003 formula and the proposed version is not noticeable).